NONCOMMUTATIVE MARTINGALE DEVIATION AND POINCARÉ TYPE INEQUALITIES WITH APPLICATIONS

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Abstract. We prove a deviation inequality for noncommutative martingales by extending Oliveira's argument for random matrices. By integration we obtain a Burkholder type inequality with satisfactory constant. Using continuous time, we establish noncommutative Poincaré type inequalities for "nice" semigroups with a positive curvature condition. These results allow us to prove a general deviation inequality and a noncommutative transportation inequality due to Bobkov and Götze in the commutative case. To demonstrate our setting is general enough, we give various examples, including certain group von Neumann algebras, random matrices and classical diffusion processes, among others.

Introduction

In probability theory it is well known that martingale inequalities can be used to prove and extend classical inequalities such as Riesz transforms and Poincaré inequalities to larger setting. Moreover, once a true probabilistic argument has been found, it is then often easier to prove dimension free estimates. This applies in particular to Riesz transforms; see Gundy [19], Pisier [43]. The impressive work of Lust-Piquard shows that, whenever the method applies, it provides the optimal constant for Riesz transforms and Poincaré inequalities. The only drawback here is that the setup for Pisier's method is so special that requires ingenuity to establish it in every single case.

Our aim here is to establish a method which applies in a fairly general noncommutative situations. Let us first set up the framework. Let $\mathcal{N}$ be a finite von Neumann algebra equipped with a normal faithful tracial state $\tau : \mathcal{N} \to \mathbb{C}$, i.e. $\tau(1) = 1$ and $\tau(xy) = \tau(yx)$. Let $(\mathcal{N}_k)_{k=1,\cdots,n} \subset \mathcal{N}$ be a filtration of von Neumann subalgebras with conditional expectation $E_k : \mathcal{N} \to \mathcal{N}_k$. For a martingale sequence (x_k) with $x_k \in \mathcal{N}_k$, we write

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$dx_k = x_k - x_{k-1}$ for the martingale differences. The starting point here is the Burkholder inequality first proved in [29]

$$\Big\| \sum_k dx_k \Big\|_p \;\le\; c(p)\Big(\Big(\sum_k \|dx_k\|_p^p \Big)^{\frac{1}{p}} + \Big\| (\sum_k dx_k^* dx_k + dx_k dx_k^*)^{1/2} \Big\|_p \Big) \,. \tag{0.1}$$

The optimal order of constant here is $c(p) \sim cp$ in the noncommutative setting, and is due to Randrianantoaninna [44]. In the commutative, dissipative setting, Barlow and Yor [4] showed a better constant in Burkholder-Davis-Gundy inequality

$$\|X_T\|_p \;\le\; c\sqrt{p}\, \|[X,X]_T^{1/2}\|_p \tag{0.2}$$

by reducing it with a time change to the case of Brownian motion. Since the nature of the Brownian motion in the noncommutative setting is so vast (see [10]), it is inconceivable that such an easy argument could work in a noncommutative situation. In fact stationarity of the Brownian motion is certainly required to perform a time change, and can no longer be guaranteed for noncommutative martingales. For many applications towards deviation inequalities it is enough to use the L_∞ norm on the right hand side of (0.2). Following Oliveira's idea [21], we are able to use Golden-Thompson inequality to prove the following result. Throughout this paper c, C and C' will always denote positive constants which may vary from line to line.

Theorem 0.1. *Let $2 \le p < \infty$ and $x_n = \sum_{k=1}^n dx_k$ be a discrete mean zero martingale. Then*

$$\|x_n\|_p \;\le\; C\sqrt{p} \|(\sum_{k=1}^n E_{k-1}(dx_k^* dx_k + dx_k dx_k^*))^{1/2}\|_\infty + Cp \sup_k \|dx_k\|_\infty \,.$$

If x_n is self-adjoint, then

$$\tau\left(1_{[t,\infty)}\,(x_n)\right) \;\le\; \exp\left(- \frac{t^2}{8\| \sum_{k=1}^n E_{k-1}(dx_k^2)\|_\infty + 4t \sup_k \|dx_k\|_\infty} \right) \,.$$

Note that in the commutative context

$$\tau(1_{[t,\infty)}(x)) \;=\; \mathrm{Prob}(x \ge t) \,.$$

In the future we will simply take this formula as a definition. The deviation inequality is a martingale version of noncommutative Bernstein inequality proved in [31]. Tropp [50] obtained better constants for tail estimate of random matrix martingales by using Lieb's concavity theorem. However, it seems Lieb's result is only applicable for commutative randomness.

Let us now indicate how to use this result in connection with curvature condition, more precisely the Γ_2-condition introduced by Bakry-Emery [3]. Here we assume that $(T_t)_{t\ge 0}$ is a semigroup of completely positive trace preserving maps on a finite von Neumann algebra $\mathcal{N}$

with infinitesimal generator $A \geq 0$. We assume in addition that (T_t) is a noncommutative diffusion process (in short (T_t) is nc-diffusion), namely that Meyer's famous gradient form

$$2\Gamma(x,x) \;=\; A(x^*)x + x^*A(x) - A(x^*x) \in L_1(N) \tag{0.3}$$

for all $x \in \mathrm{Dom}(A) \cap \mathcal{N}$ (To be more precise, for all $x \in \mathrm{Dom}(A^{1/2}) \cap \mathcal{N}$ by extension). In this paper, $\mathrm{Dom}(A)$ denotes the domain of the operator A in the underlying Hilbert space. Recall that according to [9, Section 9], we always have $\Gamma(x,x) \in \mathcal{B}^*$, where $\mathcal{B} = \mathrm{Dom}(A^{1/2}) \cap \mathcal{N}$ is a $*$-algebra by Davies and Lindsay [13, Proposition 2.8]. Thus (0.3) is a regularity assumption, which in general is much weaker than assuming that (T_t) is usual diffusion semigroup. The standard example for a nc-diffusion semigroup which is not diffusion is the Poisson semigroup on the circle. Let $\alpha > 0$ be a constant in what follows.

Theorem 0.2. *Let (T_t) be a nc-diffusion semigroup on a von Neumann algebra $\mathcal{N}$ satisfying the Γ_2-condition*

$$\tau(\Gamma(T_tx, T_tx)y) \;\leq\; Ce^{-2\alpha t}\tau(T_t\Gamma(x,x)y) \tag{0.4}$$

for all $x \in \mathrm{Dom}(A^{1/2})$ and all positive $y \in \mathcal{N}$. Then for self-adjoint x we have

$$\|x - E_{\mathrm{Fix}}(x)\|_p \;\leq\; C'\alpha^{-1/2}\min\{\sqrt{p}\,\|\Gamma(x,x)^{1/2}\|_\infty,\; p\|\Gamma(x,x)^{1/2}\|_p\}\,.$$

Here $\mathrm{Fix} = \{x : T_tx = x\}$ is the fixed point von Neumann subalgebra given by T_t and E_{Fix} the corresponding conditional expectation.

The condition (0.4) is usually formulated in the form $\Gamma_2(x,x) \geq \alpha\Gamma(x,x)$ where

$$2\Gamma_2(x,y) \;=\; \Gamma(Ax,y) + \Gamma(x,Ay) - A\Gamma(x,y)\,.$$

As in the commutative case, this result implies the deviation inequality and exponential integrability.

Corollary 0.3. *Under the hypotheses above, we have*

$$\mathrm{Prob}(x - E_{\mathrm{Fix}}x \geq t) \;\leq\; \exp\Big(-\frac{t^2}{C(\alpha)\|\Gamma(x,x)\|_\infty}\Big)\,.$$

Assume further $\tau(x) = 0$. Then

$$\tau(e^{tx}) \;\leq\; \exp\Big(c(\alpha)t^2\|\Gamma(x,x)\|_\infty\Big).$$

Quite surprisingly, these results apply to many commutative and noncommutative examples which cannot be treated with the usual commutative diffusion semigroup approach. Moreover, Bobkov and Götze's [7] application to the L_1 Wasserstein distance

$$W_1(f,h) \;=\; \sup\{|\phi_f(x) - \phi_h(x)| : x \text{ self-adjoint }, \|\Gamma(x,x)\|_\infty \leq 1\}$$

for normal state $\phi_f(x) = \tau(fx)$, $\phi_h(x) = \tau(hx)$, remains applicable in the noncommutative setting. This leads to the transportation inequality.

Corollary 0.4. *Under the assumptions above*

$$W_1(f, E_{\mathrm{Fix}} f) \ \leq \ C(\alpha)\sqrt{\tau(f \ln f)}$$

for all normal states $\phi_f(x) = \tau(fx)$.

The Wasserstein distance has been extensively studied in the noncommutative setting; see[5, 38, 45]. This probabilistic connection which provides universal upper bound given by the Entropy functional, however, seems to be new. Our definition of Wasserstein distance in the noncommutative setting is closely related to the metric used by Rieffel to define his quantum metric space. Inspiring by Connes' work in noncommutative geometry [11], Rieffel defined the metric on the state space of a $*$-algebra A

$$\rho_L(\phi, \psi) \ = \ \sup\{|\phi(a) - \psi(a)| : L(a) \leq 1, a \in A\}$$

where ϕ,ψ are states and $L(a)$ is a seminorm. For Connes' spectral triple, $L(a) = \|[D, a]\|$ where D is a self-adjoint operator; see [45] and the references therein for more details. It is an interesting question but beyond the scope of this paper to determine whether the transportation inequality is possible for ρ_L.

At this point it seems helpful to compare our approach with previous ones. Using classical diffusion theory, it is proved by Bakry and Emery [3] that the Γ_2-criterion implies the logarithmic Sobolev inequality (LSI). Bobkov and Götze [7] deduced an exponential integrability (EI) result based on a variant of LSI and showed that the EI is equivalent to the transportation inequality (TI). The relation can be illustrated by the following

$$\Gamma_2\text{-criterion} \xRightarrow{\text{diffusion}} \text{ LSI } \Rightarrow \text{ EI } \Leftrightarrow \text{ TI } .$$

We refer the reader to the lecture notes [18] for more details on this subject and its applications to random matrices. In the noncommutative setting, however, the Γ_2-criterion no longer implies LSI; see Example 3.11 below. But we can still use our L_p Poincaré inequality (L_pPI) to deduce EI and TI. In particular, this gives an alternative proof in the commutative case. Our approach is illustrated as follows

$$\Gamma_2\text{-criterion} \xRightarrow{\text{nc-diffusion}} L_p\text{PI} \Rightarrow \text{ EI } \Leftrightarrow \text{ TI } .$$

At the time of this writing, we are not sure whether this alternative is known or not in the commutative case. In addition, a simple argument shows that EI would hold (thus TI follows) provided the space has finite diameter. This gives a criterion for the validity of TI when we only have $\Gamma_2(x, x) \geq 0$ instead of $\Gamma_2(x, x) \geq \alpha\Gamma(x, x)$.

At the end of the paper we consider an algebraic version of the Γ_2-condition and say that $\Gamma_2 \ \geq \ \alpha\Gamma$ (in the form sense) if

$$[\Gamma_2(x_j, x_k)]_{j,k} \ \geq \ \alpha[\Gamma(x_j, x_k)]_{j,k}$$

for all finite families in a weakly dense A invariant algebra $\mathcal{A} \subset \mathcal{N}$. Then we collect/prove the following facts

- $\Gamma_2 \ \geq \ \Gamma$ for a suitable semigroup on group von Neumann algebra of the free group $\mathbb{F}_n$ and the noncommutative tori.
- $\Gamma_2 \ \geq \ \frac{n+2}{2n}\Gamma$ for suitable semigroups on group von Neumann algebra of the discrete Heisenberg group and the hyperfinite II_1 factor.
- Let $\mathcal{N} = L_\infty(\{-1, 1\})$ and $T_t(1) = 1$, $T_t(\varepsilon) = e^{-t}\varepsilon$ for $\varepsilon = -1$. Then $\Gamma_2 \ \geq \ \Gamma$.
- Let $\mathcal{N} = L_\infty(\{1, \cdots, n\})$ and $T_t(e^{\frac{2\pi ik}{n}}) = e^{-t(1-\delta_{k,0})}e^{\frac{2\pi ik}{n}}$. Then $\Gamma_2 \ \geq \ \frac{n+2}{2n}\Gamma$.
- $\Gamma_2 \ \geq \ \Gamma$ for all q-Gaussian random variables and the number operator.
- $\Gamma_2 \ \geq \ \alpha\Gamma$ for compact Riemannian manifolds with strictly positive Ricci curvature.
- $\Gamma_2 \ \geq \ \alpha\Gamma$ is stable under tensor products.
- $\Gamma_2 \ \geq \ \alpha\Gamma$ is stable under free products.
- $\Gamma_2 \ \geq \ \frac{n+2}{2n}\Gamma$ for a suitable semigroup on random matrices M_n.

We hope that this ample evidence that Poincaré type inequalities occur frequently in the commutative and the noncommutative setting even without assuming the strong diffusion assumption used in the Bakry-Emery theory justifies our new noncommutative theory. As special cases of our general theory, previous results obtained by Efraim/Lust-Piquard [15] and Li [35] are generalized or improved and many new inequalities are established in different contexts. For instance, the following deviation inequality for product probability spaces is an easy consequence of these examples: for $f \in L_\infty(\Omega_1 \times \cdots \times \Omega_n, \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_n)$,

$$\mathbb{P}(f - \mathbb{E}(f) \geq t) \ \leq \ \exp\Big(- \frac{ct^2}{\sum_{i=1}^n \|f - \int f d\mathbb{P}_i\|_\infty^2 + \| \int(|f|^2 - |\int f d\mathbb{P}_i|^2) d\mathbb{P}_i\|_\infty} \Big) .$$

where $\mathbb{P} = \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_n$ and $\mathbb{E}$ is the corresponding expectation operator.

The paper is organized as follows. The martingale deviation inequality and Burkholder type inequality are proved in Section 1. After recalling some results in the continuous filtrations in von Neumann algebras, we deduce two BDG type inequalities in Section 2. The Poincaré type inequalities and the transportation inequalities are proved in Section 3, which is also the most technical section. Then the group von Neumann algebras are considered in Section 4. In section 5 we prove that the Γ_2-criterion is stable under tensor products and free products with amalgamation. The general theory is applied to classical diffusion processes in Section 6.

1. Noncommutative martingale deviation inequality

Our proof of the martingale deviation inequality relies on the well known Golden-Thompson inequality. The fully general case is due to Araki [1]. The version for semifinite von Neumann algebras we used here was proved by Ruskai in [46, Theorem 4].

Lemma 1.1 (Golden-Thompson inequality)**.** *Suppose that a, b are self-adjoint operators, bounded above and that $a+b$ are essentially self-adjoint(i.e. the closure of $a+b$ is self-adjoint). Then*

$$\tau(e^{a+b}) \;\le\; \tau(e^{a/2}e^{b}e^{a/2})\,.$$

Furthermore, if $\tau(e^a)<\infty$ or $\tau(e^b)<\infty$ then

$$\tau(e^{a+b}) \;\le\; \tau(e^{a}e^{b})\,. \tag{1.1}$$

Lemma 1.2. *Let (x_k) be a self-adjoint martingale sequence with respect to the filtration $(\mathcal{N}_k, E_k)$ and $d_k := dx_k = x_k - x_{k-1}$ be the associated martingale differences such that*

i) $\tau(x_k) = x_0 = 0$; ii) $\|d_k\| \le M$; iii) $\sum_{k=1}^{n} E_{k-1}(d_k^2) \le D^2 1$.

Then

$$\tau(e^{\lambda x_n}) \;\le\; \exp[(1+\varepsilon)\lambda^2 D^2]$$

for all $\varepsilon \in (0,1]$ and all $\lambda \in [0, \sqrt{\varepsilon}/(M+M\varepsilon)]$.

Proof. We follow Oliveira's original proof for matrix martingales [21] and generalize it to the fully noncommutative setting. With the help of functional calculus, we actually have fewer technical issues. Let $\varepsilon \in (0,1]$. Put $y_n = \sum_{k=1}^{n} E_{k-1}(d_k^2)$. Then $y_n \le D^2 1$. We simply write D^2 for the operator $D^2 1 \in \mathcal{N}$ in the following. Let us first assume $M=1$. Since $e^{-((1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_n)} \le 1$, it follows from (1.1) that

$$\begin{aligned}\tau\big(e^{\lambda x_n}\big) \;&\le\; \tau\Big(\exp[\lambda x_n + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_n]\exp[-((1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_n)]\Big)\\ &\le\; \tau\Big(\exp[\lambda x_n + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_n]\Big)\,.\end{aligned}$$

Put $r_n = E_{n-1}d_n^2$. Then $y_n = y_{n-1} + r_n$. Using (1.1) again we find

$$\begin{aligned}&\tau\Big(\exp[\lambda x_n + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_n]\Big)\\ &= \tau\Big(\exp[\lambda x_{n-1} + \lambda d_n + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_{n-1} - (1+\varepsilon)\lambda^2 r_n]\Big)\\ &\le \tau\Big(\exp[\lambda d_n - (1+\varepsilon)\lambda^2 r_n]\exp[\lambda x_{n-1} + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_{n-1}]\Big)\,.\end{aligned}$$

Since $x_{n-1}, y_{n-1} \in \mathcal{N}_{n-1}$ and E_{n-1} is trace preserving, we obtain

$$\begin{aligned}&\tau\Big(\exp[\lambda d_n - (1+\varepsilon)\lambda^2 r_n]\exp[\lambda x_{n-1} + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_{n-1}]\Big)\\ &= \tau\Big(E_{n-1}[\exp(\lambda d_n - (1+\varepsilon)\lambda^2 r_n)]\exp[\lambda x_{n-1} + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_{n-1}]\Big)\,.\end{aligned} \tag{1.2}$$

We claim that $E_{k-1}[\exp(\lambda d_k - (1+\varepsilon)\lambda^2 r_k)] \le 1$ for all $k = 1, \cdots, n$ and $0 \le \lambda \le \sqrt{\varepsilon}/(1+\varepsilon)$. Indeed,

$$\|\lambda d_k - (1+\varepsilon)\lambda^2 r_k\| \le \frac{\sqrt{\varepsilon}}{1+\varepsilon} + (1+\varepsilon)\frac{\varepsilon}{(1+\varepsilon)^2} = \frac{\sqrt{\varepsilon}+\varepsilon}{1+\varepsilon} \le 1.$$

Note that $e^x \le 1+x+x^2$ for $|x| \le 1$. It follows from functional calculus that $e^A \le 1+A+A^2$ for any self-adjoint operator A with $\|A\| \le 1$. Plugging in $A = \lambda d_k - (1+\varepsilon)\lambda^2 r_k$ and using $r_k \in \mathcal{N}_{k-1}$ and $E_{k-1} d_k = 0$ we obtain

$$\begin{aligned}
&E_{k-1}[\exp(\lambda d_k - (1+\varepsilon)\lambda^2 r_k)] \\
&\le E_{k-1}[1 + \lambda d_k - (1+\varepsilon)\lambda^2 r_k + \lambda^2 d_k^2 - (1+\varepsilon)\lambda^3 d_k r_k - (1+\varepsilon)\lambda^3 r_k d_k + (1+\varepsilon)^2\lambda^4 r_k^2] \\
&= 1 - \varepsilon\lambda^2 r_k + (1+\varepsilon)^2 \lambda^4 r_k^2\,.
\end{aligned}$$

An elementary calculation shows that $\varepsilon\lambda^2 x - (1+\varepsilon)^2\lambda^4 x^2 \ge 0$ for all $x \in [0,1]$ and $\lambda \in (0, \sqrt{\varepsilon}/(1+\varepsilon)]$. Using functional calculus of r_k again, we find

$$\varepsilon\lambda^2 r_k - (1+\varepsilon)^2\lambda^4 r_k^2 \;\ge\; 0$$

which gives the claim. Combining with (1.2), we obtain

$$\begin{aligned}
&\tau\big(\exp[\lambda x_n + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_n]\big) \\
&\le \tau\big(\exp[\lambda x_{n-1} + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_{n-1}]\big).
\end{aligned}$$

Iteratively using (1.1) and the claim $n-1$ times yields

$$\tau(e^{\lambda x_n}) \;\le\; \tau(\exp[(1+\varepsilon)\lambda^2 D^2]) \;=\; \exp[(1+\varepsilon)\lambda^2 D^2]$$

which completes the proof for $M = 1$. For arbitrary x_k, considering $x_k' = x_k/M$ leads to the conclusion. □

We remark that the exponential inequality in this lemma is crucial for the proof of law of the iterated logarithms for noncommutative martingales by the second named author in [55].

Theorem 1.3. *Let (x_k) be a self-adjoint martingale sequence with respect to the filtration $(\mathcal{N}_k, E_k)$ and $d_k := dx_k = x_k - x_{k-1}$ be the associated martingale differences such that*

i) $\tau(x_k) = x_0 = 0$; ii) $\|d_k\|_\infty \le M$; iii) $\sum_{k=1}^n E_{k-1}(d_k^2) \le D^2 1$.

Then for $t \ge 0$,

$$\mathrm{Prob}\,(x_n \ge t) \;\le\; \exp\left(-\frac{t^2}{4(1+\varepsilon)D^2 + 2(1+\varepsilon)tM/\sqrt{\varepsilon}} - \frac{\sqrt{\varepsilon}Mt^3}{2(1+\varepsilon)(2\sqrt{\varepsilon}D^2+Mt)^2}\right)$$

for all $0 < \varepsilon \le 1$.

Note that if $\varepsilon = 1$ the first term in our upper bound reduces to the same estimate as Oliveira's. In fact, the first term is always dominating.

Proof. We assume $M = 1$ first. Let $\varepsilon \in (0, 1]$. By exponential Chebyshev's inequality we have $\tau\left(1_{[t,\infty)}(x_n)\right) \le e^{-\lambda t}\tau\left(e^{\lambda x_n}\right)$ for $t > 0$. It follows from Lemma 1.2 that

$$\tau\left(1_{[t,\infty)}(x_n)\right) \ \le\ \exp(-\lambda t + (1+\varepsilon)\lambda^2 D^2).$$

Now we set

$$\lambda \ =\ \frac{t}{2(1+\varepsilon)D^2 + (1+\varepsilon)t/\sqrt{\varepsilon}}$$

which is less than $\sqrt{\varepsilon}/(1+\varepsilon)$. Then,

$$-\lambda t + (1+\varepsilon)\lambda^2 D^2 \ =\ -t^2 \cdot \frac{1 + t/(\sqrt{\varepsilon}D^2)}{4(1+\varepsilon)D^2[1 + t/(2\sqrt{\varepsilon}D^2)]^2}$$

$$= -\frac{t^2}{4(1+\varepsilon)D^2[1 + t/(2\sqrt{\varepsilon}D^2)]} - \frac{\sqrt{\varepsilon}t^3}{2(1+\varepsilon)(2\sqrt{\varepsilon}D^2 + t)^2}\,.$$

Replacing t and D with t/M and D/M respectively yields the assertion. □

Similar to the classical probability theory, we have for positive $a \in \mathcal{M}$ and for all $0 < p < \infty$,

$$\|a\|_p^p \ =\ p\int_0^\infty t^{p-1}\mathrm{Prob}(a > t)dt\,. \tag{1.3}$$

From here it is routine to estimate the p-th moment of x_n using Theorem 1.3.

Proposition 1.4. *Under the assumption of Theorem 1.3, for $2 \le p < \infty$ we have*

$$\|x_n\|_p \ \le\ 2^{3/2}(1+\varepsilon)^{1/2}\sqrt{p}\Big\|\sum_{i=1}^n E_{i-1}(dx_i^2)\Big\|_\infty^{1/2} + 2^{5/2}\Big(\frac{1+\varepsilon}{\sqrt{\varepsilon}}\Big)p\sup_{i=1,\cdots,n}\|dx_i\|_\infty \tag{1.4}$$

for all $0 < \varepsilon \le 1$.

Proof. Our strategy is to integrate the first term in Theorem 1.3. The proof is the similar to that of [31, Corollary 0.3]. Note that it follows from symmetry that

$$\mathrm{Prob}\left(|x_n| \ge t\right) \ \le\ 2\exp\left(-\frac{t^2}{4(1+\varepsilon)D^2 + 2(1+\varepsilon)tM/\sqrt{\varepsilon}}\right)\,.$$

Using (1.3), we obtain

$$\frac{\|x_n\|_p^p}{2p} \ \le\ \int_0^{\frac{2\sqrt{\varepsilon}D^2}{M}} t^{p-1}\exp\left(-\frac{t^2}{8(1+\varepsilon)D^2}\right)dt + \int_{\frac{2\sqrt{\varepsilon}D^2}{M}}^\infty t^{p-1}\exp\left(-\frac{t\sqrt{\varepsilon}}{4(1+\varepsilon)M}\right)dt.$$

Let us estimate the first term on the right hand side. Using the fact that $\Gamma(x) \leq x^{x-1}$ for $x \geq 1$, we have

$$\int_0^{\frac{2\sqrt{\varepsilon}D^2}{M}} t^{p-1} \exp\left(-\frac{t^2}{8(1+\varepsilon)D^2}\right) dt \;=\; 2^{3p/2-1}(1+\varepsilon)^{p/2}D^p \int_0^{\frac{\varepsilon D^2}{2M^2(1+\varepsilon)}} r^{p/2-1}e^{-r}dr$$
$$\leq 2^{3p/2-1}(1+\varepsilon)^{p/2}D^p \int_0^\infty r^{p/2-1}e^{-r}dr \;\leq\; 2^{3p/2-1}(1+\varepsilon)^{p/2}D^p(p/2)^{p/2-1}$$
$$\leq 2^p(1+\varepsilon)^{p/2}D^p p^{p/2-1}\,.$$

For the second term on the right hand side,

$$\int_{\frac{2\sqrt{\varepsilon}D^2}{M}}^\infty t^{p-1}\exp\left(-\frac{t\sqrt{\varepsilon}}{4(1+\varepsilon)M}\right)dt$$
$$\leq 4^p\Big(\frac{1+\varepsilon}{\sqrt{\varepsilon}}\Big)^p M^p \int_0^\infty r^{p-1}e^{-r}dr \;\leq\; 4^p\Big(\frac{1+\varepsilon}{\sqrt{\varepsilon}}\Big)^p M^p p^{p-1}\,.$$

Hence, we find

$$\|x_n\|_p^p \;\leq\; 2^{p+1}(1+\varepsilon)^{p/2}D^p p^{p/2} + 2^{2p+1}\Big(\frac{1+\varepsilon}{\sqrt{\varepsilon}}\Big)^p M^p p^p.$$

This yields

$$\|x_n\|_p \;\leq\; 2^{1+1/p}(1+\varepsilon)^{1/2}D\sqrt{p} + 2^{2+1/p}\Big(\frac{1+\varepsilon}{\sqrt{\varepsilon}}\Big)Mp$$
$$\leq\; 2^{3/2}(1+\varepsilon)^{1/2}D\sqrt{p} + 2^{5/2}\Big(\frac{1+\varepsilon}{\sqrt{\varepsilon}}\Big)Mp\,.$$

Setting $D^2 = \big\|\sum_{i=1}^n E_{i-1}(dx_i^2)\big\|$ and $M = \sup_{i=1,\cdots,n}\|dx_i\|$ gives the assertion. $\square$

Another way to obtain (0.2) would be an improved Burkholder inequality for noncommutative martingales:

Problem 1.5. *Is it true that for some function $f(p)$ and constant C,*

$$\|\sum_k dx_k\|_p \;\leq\; C\sqrt{p}\|(\sum_k dx_k^*dx_k + dx_k dx_k^*)^{1/2}\|_p + f(p)\big(\sum_k \|dx_k\|_p^p\big)^{1/p} \tag{1.5}$$

holds for all noncommutative martingales.

For independent increments this has recently been proved [31]. One would actually expect $f(p) = p$. As will become clear in the following, the validity of (1.5) would improve our main results and imply a number of results in different contexts. At the time of this writing we are unable to decide whether (1.5) holds. However, the commutative case was known to be true due to the work of Pinelis [42], who attributed it to Hitczenko.

2. Noncommutative Burkholder-Davis-Gundy type inequalities

We refer the readers to [22, 24, 26] for further details about the facts mentioned in this section. Let $x = (E_1 x, \cdots, E_n x)$ be a (finite) martingale sequence with martingale differences dx_k. For $1 \le p \le \infty$, we define

$$\|x\|_{h_p^c} \; = \; \Big\| \Big(\sum_k E_{k-1}(dx_k^* dx_k) \Big)^{1/2} \Big\|_p \, , \quad \|x\|_{h_p^r} = \|x^*\|_{h_p^c} \, .$$

For $1 \le p < \infty$, we define

$$\|x\|_{h_p^d} = \Big(\sum_k \|dx_k\|_p^p \Big)^{1/p} .$$

We are going to use the continuous filtrations $(\mathcal{N}_t)_{t\ge 0} \subset \mathcal{N}$ in the following. Recall that a martingale x is said to have almost uniform (or a.u. for short) continuous path if for every $T > 0$, every $\varepsilon > 0$ there exists a projection e with $\tau(1-e) < \varepsilon$ such that the function $f_e : [0,T] \to \mathcal{N}$ given by $f_e(t) = x_t e \in \mathcal{N}$ is norm continuous. Let $\sigma = \{0 = s_0, \cdots, s_n = T\}$ be a partition of the interval $[0,T]$ and $|\sigma|$ its cardinality. Put

$$\|x\|_{h_p^c([0,T];\sigma)} \; = \; \Big\| \sum_{j=0}^{|\sigma|-1} E_{s_j} |E_{s_{j+1}} x - E_{s_j} x|^2 \Big\|_{p/2}^{1/2}, \quad 2 \le p \le \infty \, ,$$

$$\|x\|_{h_p^d([0,T];\sigma)} \; = \; \Big(\sum_{j=0}^{|\sigma|-1} \|E_{s_{j+1}} x - E_{s_j} x\|_p^p \Big)^{1/p}, \quad 2 \le p < \infty \, ,$$

and $\|x\|_{h_p^r([0,T];\sigma)} = \|x^*\|_{h_p^c([0,T];\sigma)}$. Let $\mathcal{U}$ be an ultrafilter refining the natural order given by inclusion on the set of all partitions of $[0,T]$. Let $x \in L_p(\mathcal{N})$. For $2 \le p < \infty$, we define

$$\langle x, x \rangle_T \; = \; \lim_{\sigma, \mathcal{U}} \sum_{i=0}^{|\sigma|-1} E_{s_i} |E_{s_{i+1}} x - E_{s_i} x|^2 \, .$$

Here the limit is taken in the weak* topology and it is shown in [22] that the convergence is also true in L_p norm $\|\cdot\|_{p/2}$ for all $2 < p < \infty$. We define the continuous version of h_p norms for $2 \le p < \infty$,

$$\|x\|_{h_p^c([0,T])} \; = \; \lim_{\sigma, \mathcal{U}} \|x\|_{h_p^c([0,T];\sigma)} \, ,$$

$$\|x\|_{h_p^d([0,T])} \; = \; \lim_{\sigma, \mathcal{U}} \|x\|_{h_p^d([0,T];\sigma)} \, .$$

and $\|x\|_{h_p^r([0,T])} = \|x^*\|_{h_p^c([0,T])}$ for $2 \le p < \infty$. Then for all $2 < p < \infty$

$$\|x\|_{h_p^c([0,T])} \; = \; \|\langle x, x \rangle_T\|_{p/2}^{1/2} \, . \tag{2.1}$$

A martingale x is said to be of vanishing variation if $\|x\|_{h_p^d([0,T])} = 0$ for all $T > 0$ and all $2 < p < \infty$. We also write

$$\mathrm{var}_p(x) = \|x\|_{h_p^d([0,T])}, \quad \|x\|_{L_p(\mathrm{var})} = \sup_\sigma \|d_j x\|_{L_p(\ell_1)}$$

and let $V_p(\mathcal{N})$ denote the $L_2(\mathcal{N})$ closure of $\{x \in L_p(\mathcal{N}) : \mathrm{var}_p(x) = 0\}$.

The following results are proved in [22]. For any $y \in L_p(\mathcal{N})$, we write $d_j(y) = E_{s_j} y - E_{s_{j-1}} y$.

Theorem 2.1. *Let $2 < p < \infty$ and $x \in L_p(\mathcal{N}_T)$. Then for all $\varepsilon > 0$, there exists a decomposition $x = x_1(\varepsilon) + x_2(\varepsilon)$ satisfying the following*

(1) $\mathrm{var}_p(x_1) < \varepsilon$, $x_2 \in L_p(\mathrm{var})$.
(2) *Let $P(x) = w^*\text{-}\lim_\varepsilon x_1(\varepsilon)$. Here $w^*\text{-}\lim$ denotes the weak* limit. Then $P : L_p(\mathcal{N}) \to V_p(\mathcal{N})$ is an orthogonal projection.*
(3) *$P(x) = x$ for all x with vanishing variation.*

One may take $x_1(\varepsilon) = w^*\text{-}\lim_\sigma \sum_{j=1}^{|\sigma|} d_j(d_j x 1_{[|d_j x| \le \varepsilon]})$ where 1_B is the spectral projection of $d_j x$ restricted to the Borel set B.

Lemma 2.2. *If x has a.u. continuous path, then it is of vanishing variation.*

Now let us prove the main results of this section, which can be regarded as the noncommutative version of Burkholder-Davis-Gundy type inequalities.

Theorem 2.3. *Let x be a mean 0 martingale with a.u. continuous path. Then for every $T > 0$, we have*

(1) *For $2 \le p < \infty$, if x is self-adjoint, then*

$$\|E_T x\|_p \ \le\ C\sqrt{p} \liminf_{\sigma,\mathcal{U}} \|x\|_{h_\infty^c([0,T];\sigma)} .$$

If x is not necessarily self-adjoint, then

$$\|E_T x\|_p \ \le\ C\sqrt{p} \liminf_{\sigma,\mathcal{U}} \left(\|x\|_{h_\infty^c([0,T];\sigma)} + \|x\|_{h_\infty^r([0,T];\sigma)} \right) ,$$

where we may take $C = 2\sqrt{2}$.

(2) *For all $2 \le p < \infty$,*

$$\|E_T x\|_p \ \le\ C' p \max\left\{ \|x\|_{h_p^c([0,T])},\ \|x\|_{h_p^r([0,T])} \right\} .$$

Proof. (1) First assume that x is self-adjoint and that $x \in \mathcal{N}_T$. We follow the strategy used in the proof of Theorem 2.1. Fix a partition σ of $[0,T]$. We write $h_p(\sigma)$ for $h_p([0,T];\sigma)$

in the following proof. Let $\varepsilon > 0$. We have $d_j x = d_j x 1_{[|d_j x| > \varepsilon]} + d_j x 1_{[|d_j x| \le \varepsilon]}$. Conditioning again, we obtain

$$d_j x \;=\; d_j(d_j x 1_{[|d_j x| > \varepsilon]}) + d_j(d_j x 1_{[|d_j x| \le \varepsilon]}) \,.$$

Put $x^\varepsilon_\sigma = \sum_{j=1}^{|\sigma|} d_j(d_j x 1_{[|d_j x| > \varepsilon]})$ and $y^\varepsilon_\sigma = \sum_{j=1}^{|\sigma|} d_j(d_j x 1_{[|d_j x| \le \varepsilon]})$. Then clearly

$$\sup_{j=1,\cdots,|\sigma|} \|d_j(d_j x 1_{[|d_j x| \le \varepsilon]})\|_\infty \;\le\; 2\varepsilon \,.$$

Using Proposition 1.4 with $\varepsilon = \eta$ for some fixed $0 < \eta \le 1$, we find

$$\|y^\varepsilon_\sigma - \tau(y^\varepsilon_\sigma)\|_p \;\le\; 2^{3/2}(1+\eta)^{1/2}\sqrt{p}\|y^\varepsilon_\sigma\|_{h^c_\infty(\sigma)} + 2^{7/2}\Big(\frac{1+\eta}{\sqrt{\eta}}\Big)p\varepsilon \,. \tag{2.2}$$

Note that

$$\begin{aligned} 0 \;\le\; E_{s_{j-1}}|d_j(d_j x 1_{[|d_j x| < \varepsilon]})|^2 \;&=\; E_{s_{j-1}}[(d_j x 1_{[|d_j x| < \varepsilon]})^2] - [E_{s_{j-1}}(d_j x 1_{[|d_j x| < \varepsilon]})]^2 \\ &\le\; E_{s_{j-1}}[(d_j x 1_{[|d_j x| < \varepsilon]})^2] \;\le\; E_{s_{j-1}}[(d_j x)^2] \,. \end{aligned}$$

Then we have

$$\|y^\varepsilon_\sigma\|_{h^c_\infty(\sigma)} \;=\; \Big\| \sum_{i=0}^{|\sigma|-1} E_{s_{j-1}}|d_j(d_j x 1_{[|d_j x| < \varepsilon]})|^2 \Big\|_\infty^{1/2}$$

$$\le \Big\| \sum_{i=0}^{|\sigma|-1} E_{s_{j-1}}|d_j x|^2 \Big\|_\infty^{1/2} \;=\; \|x\|_{h^c_\infty(\sigma)}.$$

According to Theorem 2.1 (in our context, $x_1(\varepsilon) = y^\varepsilon := w^*\text{-}\lim_\sigma y^\varepsilon_\sigma$) and Lemma 2.2, we have $x = w^*\text{-}\lim_{\varepsilon\to 0} y^\varepsilon$. Hence for any $\lambda_i \ge 0, \sum_{i=1}^k \lambda_i = 1$, we have

$$x \;=\; w^*\text{-}\lim_{\substack{\varepsilon_i \to 0\\ i=1,\cdots,k}} \sum_{i=1}^k \lambda_i y^{\varepsilon_i} \,.$$

Since in a Banach space the weak closure and the norm closure of a convex set are the same, by the reflexivity of $L_p(\mathcal{N})$ we can find a net x_α in the convex hull of $\{y^\varepsilon\}$ such that $x_\alpha \to x$ in $L_p(\mathcal{N})$. Therefore by sending $\varepsilon \to 0$, we deduce from (2.2) that

$$\|x\|_p \;\le\; 2\sqrt{2}(1+\eta)^{1/2}\sqrt{p}\liminf_{\sigma,\mathcal{U}} \|x\|_{h^c_\infty(\sigma)} \,,$$

for all $0 < \eta \le 1$. Sending $\eta \to 0$ yields the first assertion. If x is not self-adjoint, we write $x = \Re(x) + i\Im(x)$ where $\Re(x) = \frac{x+x^*}{2}$ and $\Im(x) = \frac{x-x^*}{2i}$. Then the second assertion follows from the self-adjoint case by triangle inequality.

(2) Since x is of vanishing variation, $\|x\|_{h^d_p([0,T])} = 0$ for all $2 < p < \infty$. Using (0.1), we have

$$\|x\|_p \;\le\; C'p(\|x\|_{h^c_p([0,T];\sigma)} + \|x\|_{h^r_p([0,T];\sigma)}) + p\|x\|_{h^d_p([0,T];\sigma)} \,.$$

Taking limits on the right hand side yields the assertion for $2 < p < \infty$. The case $p = 2$ is proved by sending $p \downarrow 2$. $\square$

3. Poincaré type inequalities and applications

3.1. **Poincaré type inequalities.** Let $(T_t)_{t\geq 0}$ be a semigroup of operators acting on a finite von Neumann algebra $(\mathcal{N}, \tau)$ where $\tau(1) = 1$. Following [24, 25] we say (T_t) is a *standard* semigroup if it satisfies the following assumptions:

(1) Every T_t is a normal completely positive map on $\mathcal{N}$ such that $T_t(1) = 1$;
(2) Every T_t is self-adjoint, i.e. $\tau(T_t(x)y) = \tau(xT_t(y))$ for all $x, y \in \mathcal{N}$.
(3) The family (T_t) is weak* continuous, i.e. $\lim_{t\to 0} T_t f = f$ with respect to the strong operator topology in $\mathcal{N}$ for any $f \in \mathcal{N}$.

The assumption (3) is equivalent to that (T_t) is a strongly continuous semigroup on $L_2(\mathcal{N}, \tau)$. By [13], (T_t) extends to a strongly continuous contraction semigroup on $L_p(\mathcal{N})$ for every $1 \leq p < \infty$ with generator A, i.e. $T_t = e^{-tA}$. Write for $1 \leq p < \infty$

$$\mathrm{Dom}_p(A) = \{f \in L_p(\mathcal{N}) : \lim_{t\to 0}(f - T_t f)/t \text{ converges in } L_p(\mathcal{N})\}.$$

Then the classical semigroup theory asserts that $\mathrm{Dom}_p(A)$ is dense in $L_p(\mathcal{N})$ and that if $x \in \mathrm{Dom}_p(A)$ then $T_t x \in \mathrm{Dom}_p(A)$. We also denote $\mathrm{Dom}(A) = \mathrm{Dom}_2(A)$. Note that A is a positive operator on $L_2(\mathcal{N}, \tau)$. The standard assumptions also imply that $\tau(T_t x) = \tau(x)$ and thus T_t's are faithful. In addition, T_t is a contraction on $\mathcal{N}$. Indeed, for $x \in \mathcal{N}$, we have

$$\|T_t x\|_\infty = \sup_{\|y\|_1\leq 1} |\tau((T_t x)y)| = \sup_{\|y\|_1\leq 1} |\tau(x(T_t y))| \leq \sup_{\|y\|_1\leq 1} \|T_t y\|_1 \|x\|_\infty \leq \|x\|_\infty .$$

Recall that T_t is said to admit a Markov dilation if

(H1) there exists a larger finite von Neumann algebra $\mathcal{M}$ and a family $\pi_t : \mathcal{N} \to \mathcal{M}$ of trace preserving $*$-homomorphism;
(H2) there is an increasing filtration $(\mathcal{M}_{s]})_{0\leq s<\infty}$ with $\pi_r(x) \in \mathcal{M}_{s]}$ for all $r < s$ such that $E_{s]}(\pi_t(x)) = \pi_s(T_{t-s}x)$ for all $s < t$ and $x \in \mathcal{N}$.

T_t is said to admit a reversed Markov dilation if condition (H1) holds and

(H2') there is a decreasing filtration $(\mathcal{M}_{[s})_{0\leq s<\infty}$ with $\pi_r(x) \in \mathcal{M}_{[s}$ for all $r > s$ such that $E_{[s}(\pi_t(x)) = \pi_s(T_{s-t}x)$ for all $t < s$ and $x \in \mathcal{N}$.

Here we have $\mathcal{M}_{t]} = E_{t]}(\mathcal{M})$ and $\mathcal{M}_{[t} = E_{[t}(\mathcal{M})$. For elements $x, y \in \mathrm{Dom}(A)$ we may define the gradient form, which is called Meyer's "carré du champ" in the commutative

theory,

$$2\Gamma(x,y) = A(x^*)y + x^*A(y) - A(x^*y)$$

and for $x, y \in \mathrm{Dom}(A^2)$ the second order gradient

$$2\Gamma_2(x,y) = \Gamma(Ax,y) + \Gamma(x,Ay) - A\Gamma(x,y).$$

Recall that (T_t) is called a noncommutative diffusion (or nc-diffusion for short) semigroup if $\Gamma(x,x) \in L_1(\mathcal{N})$ for all $x \in \mathrm{Dom}(A^{1/2})$. If (T_t) is nc-diffusion, then $\Gamma(x,x) \in L_1(\mathcal{N})$ is well-defined for $x \in \mathrm{Dom}(A^{1/2})$ by extension. By duality, $\Gamma(x,x) \in L_p(\mathcal{N})$ for $1 \le p < \infty$ if and only if there exists $C > 0$ such that $|\tau(\Gamma(x,x)y)| \le C\|y\|_{p'}$ for all y and $1/p + 1/p' = 1$.

We will use the following crucial results proved by Junge, Ricard, and Shlyakhtenko in [27], which is a noncommutative version of the Stroock-Varadhan martingale problem.

Theorem 3.1. *Suppose that $(T_t)_{t\ge 0}$ is a standard nc-diffusion semigroup. Then*

(1) *T_t admits a Markov dilation (π_t) with a.u. continuous path, i.e. in addition to (H1) and (H2), for all $x \in \mathrm{Dom}(A)$,*

$$m_t(x) := \pi_t(x) + \int_0^t \pi_s(AT_s x)ds$$

is a martingale with a.u. continuous path.

(2) *T_t admits a reversed Markov dilation $(\widetilde{\pi}_t)$ with a.u. continuous path, i.e. in addition to (H1) and (H2'), for all $x \in \mathrm{Dom}(A)$ and all $S > 0$,*

$$\widetilde{m}_s(x) := \widetilde{\pi}_s(T_s(x)), \quad 0 \le s \le S$$

is a (reversed) martingale with a.u. continuous path.

Remark 3.2. Let $2 \le p < \infty$. For the purpose of our main result, we extend the theorem to $x \in \mathrm{Dom}(A^{1/2})$. Indeed, since $\mathrm{Dom}(A^{1/2}) \cap \mathcal{N}$ is a $*$-subalgebra of $\mathcal{N}$ by [13] and $\mathrm{Dom}(A)$ is dense in $L_2(\mathcal{N})$, there exists a sequence $(x_n) \in \mathrm{Dom}(A)$ such that $\lim_{n\to\infty} \|x_n - x\|_2 = 0$. But $E_{[r}(\widetilde{\pi}_s T_s(x_n)) = \widetilde{\pi}_r T_r(x_n)$ for $s < r$. Taking limits on both sides, we find $E_{[r}(\widetilde{\pi}_s T_s(x)) = \widetilde{\pi}_r T_r(x)$ in $L_2(\mathcal{N})$. According to [36], the set of a.u. continuous path martingales is closed in $L_2(\mathcal{N})$. Hence $\widetilde{\pi}_s T_s(x)$ has a.u. continuous path. Similar argument applies to the forward martingales, but we only need the reversed martingales in this paper.

Put $L_p^0(\mathcal{N}) = \{x \in L_p(\mathcal{N}) : \lim_{t\to\infty} T_t x = 0\}$ for $1 \le p \le \infty$. Here the limit is taken with respect to $\|\cdot\|_{L_p(\mathcal{N})}$ for $1 < p < \infty$ and with respect to the weak* topology for $p = 1, \infty$. Let $\mathrm{Fix} = \{x \in \mathcal{N} : T_t x = x\}$. Then it was shown in [30] that Fix is a von Neumann subalgebra and $\mathrm{Fix}^\perp = L_\infty^0(\mathcal{N})$. Denote by $E_{\mathrm{Fix}} : \mathcal{N} \to \mathrm{Fix}$ the conditional

expectation which extends to a contraction on $L_p(\mathcal{N})$. Then for all $x \in L_p(\mathcal{N})$ we have $x - E_{\mathrm{Fix}}x \in L_p^0(\mathcal{N})$ and $L_p^0(\mathcal{N})$ is a complemented subspace of $L_p(\mathcal{N})$.

Lemma 3.3. *Let $2 \le p < \infty$ and $(T_t)_{t\ge 0}$ be a standard nc-diffusion semigroup. Then for all $0 \le s < t \le \infty$, and $x \in \mathrm{Dom}(A^{1/2}) \cap L_p^0(\mathcal{N})$ with $\Gamma(T_r x, T_r x)$ uniformly bounded for $r \ge 0$ in $L_p(\mathcal{N})$ we have*

$$\|\widetilde{m}(x)\|_{h_p^c([s,t])} \;=\; 2\left\|\int_s^t \widetilde{\pi}_r(\Gamma(T_r x, T_r x))dr\right\|_{p/2}^{1/2} .$$

Proof. In the proof we write $m_t = \widetilde{m}_t(x)$ for simplicity. By Theorem 3.1, Remark 3.2 and Lemma 2.2, $\mathrm{var}_p(m) = 0$ for all $2 < p < \infty$. (2.1) implies for $2 < p < \infty$,

$$\|m\|_{h_p^c([s,t])} \;=\; \|\langle m,m\rangle_t - \langle m,m\rangle_s\|_{p/2}^{1/2} .$$

It follows from [24, Lemma 2.4.1] and uniform boundedness that

$$\langle m,m\rangle_s - \langle m,m\rangle_t \;=\; 2\int_s^t \widetilde{\pi}_r(\Gamma(T_r x, T_r x))dr .$$

Here the integral when $t = \infty$ is well-defined for $x \in L_p^0(\mathcal{N})$ according to [24, Proposition 2.4.3]. This gives the assertion for $2 < p < \infty$. The case $p = 2$ follows by sending $p \downarrow 2$. $\square$

We are now ready to state our main result of this section.

Theorem 3.4. *Suppose $2 \le p < \infty$. Let $T_t = e^{-tA}$ be a standard nc-diffusion semigroup and Γ the gradient form associated with A. Assume $x \in L_p(\mathcal{N}) \cap \mathrm{Dom}(A^{1/2})$ satisfies*

$$\tau(y\Gamma(T_t x, T_t x)) \;\le\; e^{-2\alpha t}\tau(yT_t\Gamma(x,x)), \quad y \in \mathcal{N}, y \ge 0 , \tag{3.1}$$

for some $\alpha > 0$. Then we have the following Poincaré type inequalities

$$\|x - E_{\mathrm{Fix}}x\|_p \;\le\; C\sqrt{p/\alpha}\max\{\|\Gamma(x,x)^{1/2}\|_\infty, \Gamma(x^*,x^*)^{1/2}\|_\infty\}, \tag{3.2}$$

$$\|x - E_{\mathrm{Fix}}x\|_p \;\le\; C'\alpha^{-1/2}p\max\{\|\Gamma(x,x)^{1/2}\|_p, \Gamma(x^*,x^*)^{1/2}\|_p\}, \tag{3.3}$$

where we can take $C = 8$ and $C = 4$ if x is self-adjoint.

Proof. First assume $2 < p < \infty$. Notice that $E_{\mathrm{Fix}}x$ is in the multiplicative domain of A. Then $\Gamma(x,x) = \Gamma(x - E_{\mathrm{Fix}}x, x - E_{\mathrm{Fix}}x)$. Without loss of generality we may assume $x \in L_p^0(\mathcal{N})$, which implies $\tau(x) = \lim_{t\to\infty}\tau(T_t x) = 0$. Fix a constant $0 < M < \infty$ and consider the reversed martingale $\widetilde{m}_t(x)$ in Theorem 3.1 for $t \in [0, M]$. By Theorem 2.3 (applying to reversed martingales), noticing that $\widetilde{m}_0(x) = \widetilde{\pi}_0(x) = x$, we have

$$\begin{aligned}
&\|\widetilde{m}_0 - E_{[M}(\widetilde{m}_0)\|_p \\
&\le C\sqrt{p}\liminf_{\sigma,\mathcal{U}}\left(\|\widetilde{m}_0 - E_{[M}(\widetilde{m}_0)\|_{h_\infty^c([0,M];\sigma)} + \|\widetilde{m}_0 - E_{[M}(\widetilde{m}_0)\|_{h_\infty^r([0,M];\sigma)}\right) .
\end{aligned}$$

Using the reversed Markov dilation, we find (see [24, (2.12)])

$$
\begin{aligned}
\|\widetilde{m}_0 - E_{[M}(\widetilde{m}_0)\|_{h^c_\infty([0,M];\sigma)} &= \Big\| \sum_{j=0}^{|\sigma|-1} E_{[s_{j+1}} |\widetilde{m}_{s_j}(x) - \widetilde{m}_{s_{j+1}}(x)|^2 \Big\|_\infty^{1/2} \\
&= 2\Big\| \sum_{j=0}^{|\sigma|-1} \int_{s_j}^{s_{j+1}} E_{[s_{j+1}} \widetilde{\pi}_r(\Gamma(T_r x, T_r x)) dr \Big\|_\infty^{1/2} .
\end{aligned} \tag{3.4}
$$

Since $E_{[s_{j+1}}$ and $\widetilde{\pi}_r$ are contractions, we deduce from (3.1) that

$$
\|\widetilde{m}_0 - E_{[M}(\widetilde{m}_0)\|_{h^c_\infty([0,M];\sigma)} \le 2\Big(\sum_{j=0}^{|\sigma|-1} \int_{s_j}^{s_{j+1}} \|\Gamma(T_r x, T_r x)\|_\infty dr \Big)^{1/2}
$$

$$
\begin{aligned}
&= 2\Big(\sum_{j=0}^{|\sigma|-1} \int_{s_j}^{s_{j+1}} \sup_{y\ge 0, y\in\mathcal{N}, \|y\|_1\le 1} \tau(y\Gamma(T_r x, T_r x)) dr \Big)^{1/2} \\
&\le 2\Big(\sum_{j=0}^{|\sigma|-1} \int_{s_j}^{s_{j+1}} e^{-2\alpha r} \sup_{y\ge 0, y\in\mathcal{N}, \|y\|_1\le 1} \tau(y T_r \Gamma(x,x)) dr \Big)^{1/2} \\
&= 2\Big(\sum_{j=0}^{|\sigma|-1} \int_{s_j}^{s_{j+1}} e^{-2\alpha r} \|T_r\Gamma(x,x)\|_\infty dr \Big)^{1/2} \le 2\Big(\int_0^M e^{-2\alpha r} \|\Gamma(x,x)\|_\infty dr \Big)^{1/2} \\
&\le 2\Big(\frac{1}{2\alpha} - \frac{1}{2\alpha e^{2\alpha M}} \Big)^{1/2} \|\Gamma(x,x)\|_\infty^{1/2} .
\end{aligned}
$$

Similarly,

$$
\|\widetilde{m}_0 - E_{[M}(\widetilde{m}_0)\|_{h^r_\infty([0,M];\sigma)} \le 2\Big(\frac{1}{2\alpha} - \frac{1}{2\alpha e^{2\alpha M}} \Big)^{1/2} \|\Gamma(x^*,x^*)\|_\infty^{1/2} .
$$

Hence we have

$$
\|\widetilde{m}_0 - E_{[M}(\widetilde{m}_0)\|_p \le 2C\sqrt{p}\Big(\frac{1}{2\alpha} - \frac{1}{2\alpha e^{2\alpha M}} \Big)^{1/2} (\|\Gamma(x,x)\|_\infty^{1/2} + \|\Gamma(x^*,x^*)\|_\infty^{1/2}) .
$$

By the reversed Markov dilation,

$$
\|E_{[M}(\widetilde{m}_0)\|_p = \|E_{[M}(\widetilde{\pi}_0(x))\|_p = \|\pi_M T_M x\|_p \le \|T_M x\|_p .
$$

Note that $\lim_{M\to\infty} \|T_M x\|_p = 0$ and that $\|x\|_p \le \|\widetilde{m}_0 - E_{[M}(\widetilde{m}_0)\|_p + \|E_{[M}(\widetilde{m}_0)\|_p$. Sending $M \to \infty$ gives the first assertion for $2 < p < \infty$. Sending $p \downarrow 2$ gives the case $p = 2$.

For (3.3), note that (3.1) implies $\Gamma(T_t x, T_t x)$ is uniformly bounded in $L_p(\mathcal{N})$. Then Theorem 2.3 and Lemma 3.3 imply that for $M > 0$, $2 < p < \infty$, and $x \in \mathrm{Dom}(A^{1/2})$, we have

$$
\|\widetilde{m}_0 - E_{[M}(\widetilde{m}_0)\|_p
$$

$$\leq 2C'p \max \left\{ \Big\| \int_0^M \widetilde{\pi}_r(\Gamma(T_r x, T_r x)) dr \Big\|_{p/2}^{1/2}, \ \Big\| \int_0^M \widetilde{\pi}_r(\Gamma(T_r x^*, T_r x^*)) dr \Big\|_{p/2}^{1/2} \right\} .$$

Similar to the above argument, (3.1) yields

$$\Big\| \int_0^M \widetilde{\pi}_r(\Gamma(T_r x, T_r x)) dr \Big\|_{p/2}^{1/2} \leq \Big(\frac{1}{2\alpha} - \frac{1}{2\alpha e^{2\alpha M}} \Big)^{1/2} \| \Gamma(x,x) \|_{p/2}^{1/2} .$$

The rest of proof is the same as that of the first assertion. □

If A has a spectral gap, we can deduce the second inequality (3.3) from the main result of [24] on the noncommutative Riesz transform. However, we have explicit order p here. So far as we know, no previous method has achieved the order $\sqrt{p}$ in the first inequality in the noncommutative setting.

Remark 3.5. In fact, if (3.1) holds for all $x \in \mathrm{Dom}(A^{1/2})$, then T_t is a nc-diffusion semigroup. Indeed, it was proved in [27] that $T_t \Gamma(x,x) \in L_1(\mathcal{N})$ for $t > 0$. Then (3.1) implies that $\Gamma(T_t x, T_t x) \in L_1(\mathcal{N})$. Taking limit gives $\Gamma(x,x) \in L_1(\mathcal{N})$.

The condition (3.1) is not convenient to check. In practice, we may pose stronger assumptions which are easy to verify. The following lemma is of course well known in the commutative case.

Lemma 3.6. *Let $T_t = e^{-tA}$ be a standard nc-diffusion semigroup. Let $x \in \mathcal{N}$ be such that $\Gamma_2(x,x)$ is well-defined. Then $\Gamma_2(x,x) \geq \alpha \Gamma(x,x)$ implies* (3.1).

Proof. Let $\widetilde{T}_t = e^{2\alpha t} T_t$. Consider the function

$$f(s) = \widetilde{T}_{t-s} \Gamma(\widetilde{T}_s x, \widetilde{T}_s x) = e^{2\alpha(t+s)} T_{t-s} \Gamma(T_s x, T_s x).$$

Due to the assumption, $f(s)$ is differentiable. Since T_t is positive, $\alpha T_t \Gamma(x,x) \leq T_t \Gamma_2(x,x)$. Then

$$\begin{aligned} f'(s) &= 2\alpha e^{2\alpha(t+s)} T_{t-s} \Gamma(T_s x, T_s x) + e^{2\alpha(t+s)} T_{t-s} A \Gamma(T_s x, T_s x) \\ &\quad - e^{2\alpha(t+s)} T_{t-s} [\Gamma(A T_s x, T_s x) + \Gamma(T_s x, A T_s x)] \\ &= 2\alpha e^{2\alpha(t+s)} T_{t-s} \Gamma(T_s x, T_s x) - 2 e^{2\alpha(t+s)} T_{t-s} \Gamma_2(T_s x, T_s x) \ \leq \ 0 \end{aligned}$$

for all $0 < s < t$. We have by continuity $\Gamma(\widetilde{T}_t x, \widetilde{T}_t x) = f(t) \ \leq \ f(0) = \widetilde{T}_t \Gamma(x,x)$, or $\Gamma(T_t x, T_t x) \leq e^{-2\alpha t} T_t \Gamma(x,x)$ which implies (3.1). □

For the purpose of future development, let us recall the definition of positive forms. Suppose $\Theta : \mathcal{N} \times \mathcal{N} \to L_1(\mathcal{N}, \tau)$ is a sesquilinear form whose domain is a weakly dense $*$-subalgebra $\mathrm{Dom}(\Theta)$ such that $1 \in \mathrm{Dom}(\Theta)$. In this paper, we follow the convention that a sesquilinear form is conjugate linear in the first component. Θ is said to be positive if for

all $n \in \mathbb{N}$, $x_1, \cdots, x_n \in \mathrm{Dom}(\Theta)$, $(\Theta(x_i, x_j))_{i,j=1}^n$ is positive in $M_n(L_1(\mathcal{N}))$. Given another sesquilinear form Φ, $\Theta \ge \Phi$ if $\Theta - \Phi \ge 0$. We refer the readers to [41, 47] for more details. For any $n \in \mathbb{N}$, and any $a_1, \cdots, a_n \in \mathrm{Dom}(A)$ the $n \times n$ matrix $(\Gamma(a_i, a_j))_{i,j=1}^n$ with entries in $\mathcal{N}$ is positive in $M_n(\mathcal{N})$. The following useful fact was due to Peterson [41]; see also [47] for the implication "$\Rightarrow$".

Theorem 3.7. *Let (T_t) be a strongly continuous semigroup on $L_2(\mathcal{N})$. Then (T_t) is a completely positive semigroup if and only if Γ is a positive form.*

As in the commutative case, the domain of Γ and Γ_2 is a delicate issue. Theorem 3.4 avoided this difficulty by considering individual element. In many cases we are interested in the Poincaré type inequalities for the whole space. Our next result is meant for this purpose.

Corollary 3.8. *Let $T_t = e^{-tA}$ be a standard nc-diffusion semigroup. Suppose that there exists a weakly dense self-adjoint subalgebra $\mathcal{A} \subset \mathcal{N}$ such that*
i) $A(\mathcal{A}) \subset \mathcal{A}$; ii) $T_t(\mathcal{A}) \subset \mathcal{A}$; iii) $\mathcal{A}$ is dense in $\mathrm{Dom}(A^{1/2})$ in the graph norm of $A^{1/2}$.
Assume $\Gamma_2(x,x) \ge \alpha\Gamma(x,x)$ for some $\alpha > 0$ and for all $x \in \mathcal{A}$. Then (3.1) holds for all $x \in \mathrm{Dom}(A^{1/2})$. Moreover, all $x \in L_p(\mathcal{N})$ satisfies (3.2) and (3.3).

Proof. For $x \in \mathrm{Dom}(A^{1/2})$ we deduce from assumption iii) that there exist $(x_n) \subset \mathcal{A}$ with $\Gamma_2(x_n, x_n) \ge \alpha\Gamma(x_n, x_n)$ such that $\|x_n - x\|_2 \to 0$ and $\|A^{1/2}x_n - A^{1/2}x\|_2 \to 0$ as $n \to \infty$. By [9, Section 9], we have $\|\Gamma(x,x)\|_1 = \langle A^{1/2}x, A^{1/2}x\rangle_{L_2(\mathcal{N})}$. Since Γ is a complete positive form and τ is also complete positive, we have for $x, y \in \mathrm{Dom}(A^{1/2})$,

$$\begin{pmatrix} \tau(\Gamma(x,x)) & \tau(\Gamma(x,y)) \\ \tau(\Gamma(y,x)) & \tau(\Gamma(y,y)) \end{pmatrix} \ge 0 \,.$$

Note that $\Gamma(x,y)^* = \Gamma(y,x)$. Then we have $\|\Gamma(x,y)\|_1^2 \le \|\Gamma(x,x)\|_1 \|\Gamma(y,y)\|_1$. It follows that

$$\begin{aligned} \|\Gamma(x_n,x_n) - \Gamma(x,x)\|_1 \ &\le \ \|\Gamma(x_n - x, x_n - x)\|_1 + 2\|\Gamma(x_n - x, x)\|_1 \\ \le \langle A^{1/2}(x_n - x), A^{1/2}(x_n - x)\rangle_{L_2(\mathcal{N},\tau)} &+ 2\langle A^{1/2}(x_n - x), A^{1/2}(x_n - x)\rangle_{L_2(\mathcal{N},\tau)}^{1/2} \|\Gamma(x,x)\|_1^{1/2} \,. \end{aligned}$$

Hence $\lim_{n\to\infty} \Gamma(x_n, x_n) = \Gamma(x,x)$ in $L_1(\mathcal{N})$. Notice that T_t and $A^{1/2}$ commute. Then for all $t \ge 0$ and $x \in \mathrm{Dom}(A^{1/2})$, $T_t x \in \mathrm{Dom}(A^{1/2})$ and a similar argument as above gives that $\lim_{n\to\infty} \Gamma(T_t x_n, T_t x_n) = \Gamma(T_t x, T_t x)$ in $L_1(\mathcal{N})$. Since Lemma 3.6 implies

$$\tau(y\Gamma(T_t x_n, T_t x_n)) \le e^{-2\alpha t}\tau(yT_t\Gamma(x_n, x_n))$$

for all $y \in \mathcal{N}, y \ge 0$, sending $n \to \infty$ on both sides yields the first assertion. For the "moreover" part, note that we only need to prove (3.2) and (3.3) for

$$\max\{\|\Gamma(x,x)^{1/2}\|_p, \|\Gamma(x^*,x^*)^{1/2}\|_p\} < \infty.$$

Recall that $\Gamma(x,x)$ is understood as the weak* limit of $\Gamma^{A_\varepsilon}(x,x)$ where $A_\varepsilon = (I+\varepsilon A)^{-1}A$ (see [9, (3.2)]). If this limit exits in L_p for $p>1$, then $\tau(\Gamma(x,x))$ is finite and hence $x \in \mathrm{Dom}(A^{1/2})$. The individual result Theorem 3.4 then comes into play and completes the proof. □

The condition $\Gamma_2(x,x) \geq \alpha\Gamma(x,x)$ we posed here is usually called the curvature condition or Γ_2-criterion. $\max\{\|\Gamma(x,x)\|_\infty,\ \|\Gamma(x^*,x^*)\|_\infty\}$ is the so-called Lipschitz norm in the commutative theory.

Remark 3.9. If we are only interested in classical diffusion processes, we can use the classical BDG inequality (0.2) for continuous path martingales with the best order $\sqrt{p}$ proved in [4]. Following verbatim the strategy developed in this section–especially replacing Theorem 3.1 by the well-known Stroock-Varadhan (reversed) martingale problem (see [48])–we can improve the order to $\sqrt{p}$ in (3.3), which is stronger than both inequalities obtained in Theorem 3.4. This is the case for the classical diffusion processes considered later in this paper. The more satisfactory inequality in the general noncommutative setting would follow from (1.5) if it were true. In the commutative case, the relationship among BDG inequality, Γ_2-criterion, and Poincaré inequality is probably known to experts.

Bakry and Emery showed [3] in the commutative case that Γ_2-criterion implies the LSI. Our first example is very simple, but it clarifies that such implication is not true any more in the general noncommutative setting. The following generalized Schwartz inequality is called Choi's inequality.

Lemma 3.10. *Let $\phi : \mathcal{M} \to \mathcal{N}$ be a contractive completely positive map between von Neumann algebras. Then $[\phi(x_i^* x_j)] \geq [\phi(x_i^*)\phi(x_j)]$ for any $n \in \mathbb{N}$ and any $x_1, \cdots, x_n \in \mathcal{M}$.*

Example 3.11 (Conditional expectation)**.** Let $E : \mathcal{M} \to \mathcal{N}$ be the conditional expectation and $A = I - E$. For $x, y \in \mathcal{M}$, a calculation gives

$$2\Gamma(x,y) \;=\; x^*y - E(x^*)y - x^*E(y) + E(x^*y)\,.$$

By Lemma 3.10, $2[\Gamma(x_i,x_j)] \geq [(x_i - Ex_i)^*(x_j - Ex_j)] \geq 0$ for $x_1, \cdots, x_n \in \mathcal{M}$. We deduce from Theorem 3.7 that A generates a completely positive semigroup $T_t = e^{-tA}$ acting on $\mathcal{M}$. It is easy to check T_t is a standard nc-diffusion semigroup. Let Γ, Γ_2 be the gradient forms associated to T_t.

Proposition 3.12. $\Gamma_2 \geq \frac{1}{2}\Gamma$ *in* $\mathcal{M}$.

Proof. Note that $AE = EA = 0$. We find

$$4\Gamma_2(x,y) \;=\; x^*y - E(x^*)y - x^*E(y) - 2E(x^*)E(y) + 3E(x^*y)\,.$$

Hence $(\Gamma_2-\frac{1}{2}\Gamma)(x,y)=\frac{1}{2}(E(x^*y)-E(x^*)E(y))$. Since E is contractive completely positive, it follows from Lemma 3.10 that $\Gamma_2-\frac{1}{2}\Gamma$ is a positive form. $\square$

The logarithmic Sobolev inequality fails, however. Indeed, the LSI reads as follows in this case: for $x\geq 0$,

$$\tau(x^2\ln(|x|/\|x\|_2)) \;\leq\; C(\|x\|_2^2-\frac{1}{2}(\tau(E(x^*)x)+\tau(x^*E(x)))\ .$$

It is easy to see this is not true.

3.2. **Deviation and transportation inequalities.** As is well-known, the Poincaré inequality with constant $\sqrt{p}$ implies the concentration phenomenon. We are going to prove a noncommutative version of exponential integrability due to Bobkov and Götze [7] in the commutative case. The following variant was due to Efraim and Lust-Piquard in the case of Walsh system.

Corollary 3.13. *Under the assumptions of Theorem 3.4, we have*

$$\tau(e^{|x-E_{\mathrm{Fix}}x|}) \;\leq\; 2\exp\Big(\frac{C}{\alpha}\max\{\|\Gamma(x,x)\|_\infty,\|\Gamma(x^*,x^*)\|_\infty\}\Big)\ , \tag{3.5}$$

and for $t>0$

$$\mathrm{Prob}(|x-E_{\mathrm{Fix}}x|\geq t) \;\leq\; 2\exp\Big(-\frac{\alpha t^2}{4C\max\{\|\Gamma(x,x)\|_\infty,\|\Gamma(x^*,x^*)\|_\infty\}}\Big)\ . \tag{3.6}$$

We may take $C=32e$ in general and $C=8e$ for x self-adjoint.

Proof. We follow the proof in the commutative case, see [15, Corollary 4.1 and 4.2]. Since $\Gamma(x,x)=\Gamma(x-E_{\mathrm{Fix}}x,x-E_{\mathrm{Fix}}x)$, we may assume $E_{\mathrm{Fix}}(x)=0$. Put $M=\max\{\|\Gamma(x,x)^{1/2}\|_\infty,\Gamma(x^*,x^*)^{1/2}\|_\infty\}$. Note that $\frac{k^k}{(2k-1)!!}\leq\left(\frac{e}{2}\right)^k$ for all $k\in\mathbb{N}$. By functional calculus and (3.2),

$$\begin{aligned}\frac{1}{2}\tau(e^{|x|}) \;&\leq\; \tau(\cosh x)=1+\sum_{k=1}^\infty\frac{1}{(2k)!}\|x\|_{2k}^{2k}\\ &\leq\; 1+\sum_{k=1}^\infty\frac{C^{2k}(2k)^k}{\alpha^k(2k)!}M^{2k} \;\leq\; 1+\sum_{k=1}^\infty\frac{k^k(CM)^{2k}}{\alpha^k k!(2k-1)!!}\\ &\leq\; 1+\sum_{k=1}^\infty\frac{(e/2)^k(CM)^{2k}}{\alpha^k k!} \;=\; \exp\Big(\frac{eC^2M^2}{2\alpha}\Big)\ .\end{aligned}$$

We have proved the first assertion for $C=32e$ and we can take $C=8e$ if x is self-adjoint. For the second inequality, we deduce from Chebyshev inequality that

$$\tau(1_{[t,\infty)}(|x|)) \;\leq\; e^{-\lambda t}\tau(e^{\lambda|x|}) \;\leq\; 2e^{-\lambda t+C\lambda^2M^2/\alpha}\ .$$

Then the assertion follows from minimizing the right hand side with respect to λ. $\square$

The improvement in the situation of commutative diffusion in Remark 3.9 also gives an intermediate term in (3.5), i.e.

$$\begin{aligned}\tau(e^{|x-E_{\mathrm{Fix}}x|}) &\le 2\max\Big\{\tau\exp\Big(\frac{C'}{\alpha}\Gamma(x,x)\Big),\ \tau\exp\Big(\frac{C'}{\alpha}\Gamma(x^*,x^*)\Big)\Big\}\\ &\le 2\exp\Big(\frac{C'}{\alpha}\max\{\|\Gamma(x,x)\|_\infty,\ \Gamma(x^*,x^*)\|_\infty\}\Big).\end{aligned}$$

We do not have such an intermediate term in the fully noncommutative generality without the help of (1.5). However, it seems the Lipschitz norm is the right choice in application to the concentration inequality. In this sense, we did not lose much even if we use a larger norm. The following result is simply a one side version of Corollary 3.13. We record it here for future references.

Proposition 3.14. *Under the hypotheses of Theorem 3.4, assume further that x is self-adjoint. Then for $t\in\mathbb{R}$*

$$\tau(e^{t(x-E_{\mathrm{Fix}}x)}) \le e^{c\|\Gamma(x,x)\|_\infty t^2}, \tag{3.7}$$

and for $t>0$,

$$\mathrm{Prob}(x-E_{\mathrm{Fix}}x\ge t) \le \exp\Big(-\frac{t^2}{4c\|\Gamma(x,x)\|_\infty}\Big) \tag{3.8}$$

where the constant c only depends on α.

Proof. Again it suffices to consider (3.7) for x with $E_{\mathrm{Fix}}(x)=0$ since $\Gamma(x,x)=\Gamma(x-E_{\mathrm{Fix}}x, x-E_{\mathrm{Fix}}x)$. From the proof of (3.5), we know there exists $C>0$ such that for $t\in\mathbb{R}$

$$\tau(e^{tx}) \le \tau(e^{tx})+\tau(e^{-tx}) \le 2e^{C\|\Gamma(x,x)\|_\infty t^2/\alpha}.$$

Then for $t^2\|\Gamma(x,x)\|_\infty\ge 1$, we have $\tau(e^{tx})\le e^{(\ln 2+C/\alpha)\|\Gamma(x,x)\|_\infty t^2}$. For $t^2\|\Gamma(x,x)\|_\infty<1$,

$$\begin{aligned}\tau(e^{tx}) &= 1+\sum_{k=2}^\infty\frac{t^k\tau(x^k)}{k!} \le 1+\sum_{k=2}^\infty\frac{t^kC^kk^{k/2}\|\Gamma(x,x)\|_\infty^{k/2}}{\alpha^{k/2}k!}\\ &\le 1+c\|\Gamma(x,x)\|_\infty t^2 \le e^{c\|\Gamma(x,x)\|_\infty t^2}\end{aligned}$$

for some constant $c=c(\alpha)$ since $\tau(x)=0$ and the series converges. The second assertion follows in the same way as (3.6). $\square$

The exponential integrability result (3.7) was proved by Bobkov and Götze [7] in the commutative case by using a variant of LSI. They also deduced a transportation inequality from (3.7). We will follow their approach to obtain a noncommutative version of transportation inequality. Since LSI is not available in our noncommutative theory, our

Poincaré inequalities might be a more universal approach to the transportation inequality. Let us first define Wasserstein distance and entropy in the noncommutative setting.

Definition 3.15. *Let ρ and σ be positive τ-measurable operators (e.g. density matrices) affiliated with $(\mathcal{M},\tau)$. The noncommutative entropy of $\rho \in L_1(\mathcal{M},\tau)$ is given by*

$$\mathrm{Ent}(\rho) \;=\; \tau(\rho \ln(\rho/\tau(\rho))) \,.$$

Let ϕ and ψ be states on $\mathcal{M}$. The L_1-Wasserstein distance between ϕ and ψ is defined by

$$W_1(\phi,\psi) \;=\; \sup\{|\phi(x)-\psi(x)| : x \text{ self-adjoint }, \|\Gamma(x,x)\|_\infty \le 1\} \,.$$

The L_1-Wasserstein distance between ρ and σ is $W_1(\rho,\sigma) = W_1(\phi_\rho,\phi_\sigma)$ for $\phi_\rho(\cdot) = \tau(\cdot\rho)/\tau(\rho)$ and $\phi_\sigma(\cdot)=\tau(\cdot\sigma)/\tau(\sigma)$.

It is easy to check that W_1 is a pseudometric but may not be a metric in general. Our definition of Wasserstein distance coincides with the classical definition in the commutative case due to the Kantorovich-Rubinstein theorem, see e.g. [51, Theorem 5.10]. It is also closely related to the quantum metric in the sense of Rieffel [45]. Now we state a general fact on the relationship between conditional expectation and entropy.

Lemma 3.16. *Let $\rho \in L_1(\mathcal{M},\tau)$ with $\rho\ge 0$ and $\tau(\rho)=1$ and $E:\mathcal{M}\to\mathcal{N}$ the conditional expectation onto subalgebra $\mathcal{N}$. Then*

$$\tau(E\rho \ln E\rho) \;\le\; \tau(\rho\ln\rho) \,.$$

Proof. Let $\rho_n = \rho 1_{[0,n]}(\rho)$. Then $\rho_n \in L_p(\mathcal{M},\tau)$ for all $p\ge 1$. It is easy to see that $\rho_n\to\rho$ in the measure topology. Notice that $\tau[(\rho_n/\tau(\rho_n))\ln(\rho_n/\tau(\rho_n))] = \lim_{p\downarrow 1}\frac{\|\rho_n/\tau(\rho_n)\|_p^p-1}{p-1}$. This yields

$$\begin{aligned}\tau\Big(\frac{E\rho_n}{\tau(\rho_n)}\ln\frac{E\rho_n}{\tau(\rho_n)}\Big) \;&=\; \lim_{p\downarrow 1}\frac{\|E\rho_n/\tau(\rho_n)\|_p^p-1}{p-1}\\ \le \lim_{p\downarrow 1}\frac{\|\rho_n/\tau(\rho_n)\|_p^p-1}{p-1} \;&=\; \tau\Big(\frac{\rho_n}{\tau(\rho_n)}\ln\frac{\rho_n}{\tau(\rho_n)}\Big)\,.\end{aligned}$$

Let μ be the distribution of ρ. Then

$$\tau\Big(\frac{\rho_n}{\tau(\rho_n)}\ln\frac{\rho_n}{\tau(\rho_n)}\Big) \;=\; \frac{1}{\tau(\rho_n)}\int_0^n x\ln x\mu(dx) - \ln\tau(\rho_n) \to \tau(\rho\ln\rho).$$

Following [16], we denote the generalized singular number of ρ by $\mu_t(\rho)$. Note that $\|E\rho_n - E\rho\|_1 \le \|\rho_n-\rho\|_1\to 0$ as $n\to\infty$. We have for every $t>0$,

$$\mu_t(E\rho_n-E\rho) \;\le\; t^{-1}\int_0^t \mu_s(E\rho_n-E\rho)ds \;\le\; t^{-1}\|E\rho_n-E\rho\|_1 \,.$$

Then $\lim_{n\to\infty}\mu_t(E\rho_n - E\rho) = 0$. By [16, Lemma 3.1], $(E\rho_n)$ converges to $E\rho$ in the measure topology. Since $0 \le E\rho_n \le E\rho$, by [16, Lemma 2.5], $\mu_t(E\rho_n) \le \mu_t(E\rho)$. We deduce from [16, Lemma 3.4] that $\lim_{n\to\infty}\mu_t(E\rho_n) = \mu_t(E\rho)$ for $t > 0$. Now consider $g(x) = x\ln x = x\ln x 1_{[1,\infty)(x)} - (-x\ln x 1_{(0,1)})$. Both functions in the decompostion are nonnegative Borel functions vanishing at the origin. It follows from [16, (3)] that $\tau(E\rho\ln E\rho) = \int_0^1 \mu_t(E\rho)\ln\mu_t(E\rho)dt$. Since for fixed $\varepsilon > 0$, $[\mu_t(E\rho_n)\ln\mu_t(E\rho_n)]_n$ is uniformly bounded on $t \in [\varepsilon, 1]$, we have

$$\begin{aligned}
\tau(E\rho\ln E\rho) \;&=\; \sup_{\varepsilon>0}\int_\varepsilon^1 \mu_t(E\rho)\ln\mu_t(E\rho)dt \\
&= \sup_{\varepsilon>0}\lim_{n\to\infty}\int_\varepsilon^1 \mu_t(E\rho_n)\ln\mu_t(E\rho_n)dt \;\le\; \limsup_{n\to\infty}\tau(E\rho_n\ln E\rho_n) \\
&\le \limsup_{n\to\infty}\tau(\rho_n\ln\rho_n)\,.
\end{aligned}$$

This completes the proof. ∎

The next result in the commutative setting is well known; see, e.g., [14, Section 6.2].

Lemma 3.17. *Let σ be a self-adjoint τ-measurable operator. Then,*

$$\ln\tau(e^\sigma) \;=\; \sup\{\tau(\rho\sigma) - \tau(\rho\ln\rho) : \rho \ge 0, \tau(\rho) = 1\}\,. \tag{3.9}$$

Therefore, for all positive $\rho \in L_1(\mathcal{M}, \tau)$

$$\mathrm{Ent}(\rho) \;=\; \sup\{\tau(\sigma\rho) : \sigma \textit{ self-adjoint}, \tau(e^\sigma) \le 1\}\,. \tag{3.10}$$

Proof. Let σ be a self-adjoint operator τ-measurable operator. Consider the von Neumann subalgebra $\mathcal{N}$ generated by $\{f(\sigma) : f : \mathbb{C} \to \mathbb{C}$ bounded measurable$\}$. Then there exists a conditional expectation $E : \mathcal{M} \to \mathcal{N}$ which can extend to a contraction $L_p(\mathcal{M}, \tau) \to L_p(\mathcal{N}, \tau)$ for all $1 \le p \le \infty$. Assume $\tau(\rho) = 1$. Then $E(\rho) \in L_1(\mathcal{N}, \tau)$. But $\mathcal{N}$ is commutative and $\tau(E(\rho)) = \tau(\rho) = 1$. After identifying τ with a probability measure denoted still by τ, we use Jensen's inequality for the measure $E(\rho)d\tau$ to deduce that

$$\tau(\sigma E(\rho)) - \tau(E(\rho)\ln E(\rho)) \;=\; \tau(\ln(e^\sigma E(\rho)^{-1})E(\rho))) \;\le\; \ln\tau(e^\sigma)\,.$$

Using Lemma 3.16 and noticing that $\tau(\sigma\rho) = \tau(\sigma E(\rho))$, we find

$$\tau(\sigma\rho) - \tau(\rho\ln\rho) \;\le\; \ln\tau(e^\sigma)\,.$$

For the reverse inequality, put $\sigma_n = \sigma 1_{(-\infty,n]}(\sigma)$ for $n \in \mathbb{N}$ where $1_{(-\infty,n]}(\sigma)$ is the spectral projection of σ. Plugging $\rho_n = e^{\sigma_n}/\tau(e^{\sigma_n})$ into the right hand side of (3.9), we have

$$\tau(\sigma\rho_n) - \tau(\rho_n\ln\rho_n) \;=\; \frac{\tau((\sigma - \sigma_n)e^{\sigma_n})}{\tau(e^{\sigma_n})} + \ln\tau(e^{\sigma_n})\,.$$

By the spectral decomposition theorem of σ, $\tau((\sigma-\sigma_n)e^{\sigma_n})\geq 0$. Then for all n we have

$$\sup\{\tau(\rho\sigma)-\tau(\rho\ln\rho):\rho\geq 0,\tau(\rho)=1\}\ \geq\ \ln\tau(e^{\sigma_n})\,.$$

By Fatou's lemma [16, Theorem 3.5] $\liminf_{n\to\infty}\ln\tau(e^{\sigma_n})\ \geq\ \ln\tau(e^{\sigma})$. This proves (3.9). For (3.10), note that $\tau(e^\sigma)\leq 1$ implies $\tau(\sigma\rho)\leq\tau(\rho\ln\rho)$ for all positive $\rho\in L_1(\mathcal{M},\tau)$ with $\tau(\rho)=1$. If $\tau(\rho)\neq 1$, we consider $\rho'=\rho/\tau(\rho)$ and find $\tau(\sigma\rho)\leq\tau(\rho\ln(\rho/\tau(\rho)))$. The equality is achieved by $\sigma=\ln\rho-\ln\tau(\rho)$. This proves the second assertion. □

Theorem 3.18. *Let $(\mathcal{M},\tau)$ be a noncommutative probability space. Then*

$$W_1(\rho,1)\ \leq\ \sqrt{2c\,\mathrm{Ent}(\rho)},\tag{3.11}$$

for all $\rho\geq 0$ with $\tau(\rho)=1$ if and only if for every self-adjoint τ-measurable operator x affiliated with $\mathcal{M}$ such that $\|\Gamma(x,x)\|_\infty\leq 1$ and $\tau(x)=0$,

$$\tau(e^{tx})\ \leq\ e^{ct^2/2},\ \textit{for all } t\in\mathbb{R}_+\,.\tag{3.12}$$

Proof. Thanks to the proceeding two lemmas, the proof is the same as that in the commutative case in [7]. We provide it here for completeness. Setting $\sigma=tx-ct^2/2$ in (3.10) and assuming (3.12) we find

$$\tau((tx-ct^2/2)\rho)\ \leq\ \mathrm{Ent}(\rho)\,.\tag{3.13}$$

Since $\tau(x)=0$ and $\tau(\rho)=1$, it follows that $\tau(x\rho-x)\leq\frac{ct}{2}+\frac{1}{t}\,\mathrm{Ent}(\rho)$. Minimizing right hand side gives

$$\tau(x\rho-x)\leq\sqrt{2c\,\mathrm{Ent}(\rho)}\,.\tag{3.14}$$

Note that $\tau(x\rho-x)=\tau(\mathring{x}\rho-\mathring{x})$ for all x where $x=\mathring{x}+\tau(x)$. Taking sup over all self-adjoint x with $\|\Gamma(x,x)\|_\infty\leq 1$ on the left hand side of (3.14) gives (3.11). For the other direction, note that (3.13) is equivalent to (3.11) by reversing the above argument. Then (3.12) follows from (3.9) by setting $\sigma=tx-ct^2/2$. □

If $\mathrm{Fix}=\mathbb{C}1$ (i.e. the system $(\mathcal{M},T_t)$ is ergodic), then combining the above theorem with (3.7), we find the transportation inequality (3.11) under the assumptions of Theorem 3.4. In fact, we even have a non-ergodic version of transportation inequality.

Corollary 3.19. *Suppose $\tau(e^{t(x-E_{\mathrm{Fix}}x)})\leq e^{ct^2}$ for any τ-measurable self-adjoint operator x affiliated to $\mathcal{M}$ such that $\|\Gamma(x,x)\|_\infty\leq 1$. Then*

$$W_1(\rho,E_{\mathrm{Fix}}\rho)\ \leq\ \sqrt{2c\,\mathrm{Ent}(\rho)}\,.\tag{3.15}$$

for all $\rho\geq 0$ with $\tau(\rho)=1$. In particular, (3.11) *holds under the additional assumption $E_{\mathrm{Fix}}\rho=1$.*

Proof. The proof modifies a little of that of Theorem 3.18. Since $\tau(\rho) = 1$, we have $\tau([t(x - E_{\mathrm{Fix}}x) - ct^2/2]\rho) \le \mathrm{Ent}(\rho)$. Then we deduce that $\tau(\rho x - \rho E_{\mathrm{Fix}}(x)) \le \sqrt{2c\,\mathrm{Ent}(\rho)}$. Since $\tau(\rho E_{\mathrm{Fix}}(x)) = \tau(E_{\mathrm{Fix}}(\rho)x)$, we have

$$\tau(\rho x - E_{\mathrm{Fix}}(\rho)x) \ \le \ \sqrt{2c\,\mathrm{Ent}(\rho)}\,.$$

Taking sup over all self-adjoint x with $\|\Gamma(x,x)\|_\infty \le 1$ gives the assertion. □

It is easy to see the assumptions are fulfilled by the hypotheses of Theorem 3.4. The point here is that even though the fixed point algebra Fix is not trivial we still have a transportation inequality although in certain situation the inequality does fail.

Remark 3.20. Let ρ be a positive operator with $\tau(\rho) = 1$. For $\rho \in \mathrm{Fix}$, define $B(\rho) = \{f \in L_1(\mathcal{N}) : E_{\mathrm{Fix}}(f) = \rho\}$. Then for $f_1, f_2 \in B(\rho)$, we have $W_1(f_1, f_2) \le W_1(f_1, \rho) + W_1(f_2, \rho) < \infty$. However, if $f_1 \in B(\rho_1)$, $f_2 \in B(\rho_2)$ and $\rho_1 \ne \rho_2$, then

$$W_1(\rho_1, \rho_2) \ge \sup\{|\tau(\rho_1 x - \rho_2 x)| : x \in \mathrm{Fix}, \|\Gamma(x,x)\|_\infty \le 1\} \ = \ \infty\,.$$

It follows that $W_1(f_1, f_2) \ge |W_1(\rho_1, \rho_2) - W_1(f_1, \rho_1) - W_1(f_2, \rho_2)| = \infty$. This yields an interesting geometric picture: operators in the same "fiber" $B(\rho)$ have finite distance between one another while operators belonging to different "fibers" have infinite distance.

The following simple result provides another way (under the assumption of finite diameter) to obtain the transportation inequality.

Corollary 3.21. *Suppose for self-adjoint $x \in \mathcal{N}$, $E_{\mathrm{Fix}}(x) = 0$ and $\|\Gamma(x,x)\|_\infty \le 1$ imply $\|x\|_\infty \le K$. Then, (3.15) holds with $c = K^2$ for all $\rho \ge 0$ such that $\tau(\rho) = 1$.*

Proof. A calculation gives $e^x - x \le e^{x^2}$. Assume $E_{\mathrm{Fix}}(x) = 0$ and $\|\Gamma(x,x)\|_\infty \le 1$. Then for $t > 0$, $\tau(e^{tx}) = \tau(e^{tx} - tx) \le \tau(e^{tK} - tK) \le e^{K^2t^2}$. The claim now follows from Corollary 3.19. □

Suppose in Theorem 3.4 we only have $\Gamma_2 \ge 0$ but not the Γ_2-condition. Junge and Mei proved in [24] as the main result

$$\|A^{1/2}x\|_p \le c(p)\max\{\|\Gamma(x,x)^{1/2}\|_p, \|\Gamma(x^*,x^*)^{1/2}\|_p\}$$

in this setting. In the same paper, they also showed [24, Theorem 1.1.7] that if

$$\|T_t : L_1^0(\mathcal{N}) \to L_\infty(\mathcal{N})\| \le Ct^{-n/2}, \tag{3.16}$$

then $\|A^{-1/2} : L_p^0(\mathcal{N}) \to L_\infty(\mathcal{N})\| \le C(n)$ for $p > n$. This gives $\|x\|_\infty \le C(n)\|A^{1/2}x\|_p$ for large p and $E_{\mathrm{Fix}}(x) = 0$. Assuming $\Gamma(x,x) \le 1$ for self-adjoint x, it follows that $\|x\|_\infty \le C(n)$. In light of Corollary 3.21, we obtain the following result.

Corollary 3.22. *Let T_t be a standard nc-diffusion semigroup acting on $\mathcal{N}$ with $\Gamma_2 \geq 0$. Then* (3.16) *with finite dimension n implies the transportation inequality* (3.15) *for all $\rho \geq 0$ such that $\tau(\rho) = 1$.*

In the commutative theory, the transportation inequality (3.11) implies isoperimetric type inequality by Marton's argument in [7]. So far it is not clear what isoperimetric inequality means in noncommutative probability. We hope to give a noncommutative analog of isoperimetric inequality.

Definition 3.23. *Let $e, f \in (\mathcal{M}, \tau)$ be projections. The distance between e and f is*

$$d(e,f) \ = \ \inf\{W_1(\phi,\psi) : \phi \text{ and } \psi \text{ are states, } s(\phi) = e, s(\psi) = f\},$$

where $s(\phi)$ is the support of ϕ.

Here our definition generalizes directly the distance of sets in the commutative theory. Thus in general d is not a metric, as in the commutative setting. Then the following result follows from the same proof as in the commutative setting given in [7].

Proposition 3.24. *Let $e, f \in (\mathcal{M}, \tau)$ be projections. Then under the assumptions of Theorem 3.4 and assuming* $\mathrm{Fix} = \mathbb{C}1$*,*

$$d(e,f) \ \leq \ \sqrt{-2c \ln \tau(e)} + \sqrt{-2c \ln \tau(f)}\,.$$

Equivalently, for every $h > \sqrt{-2c \ln \tau(e)}$ and every projection p such that $d(p,e) > h$,

$$\tau(p) \ \leq \ \exp\Big(- \frac{1}{2c}\Big(h - \sqrt{-2c\ln \tau(e)}\Big)^2\Big)\,.$$

Proof. Put $\phi_e(\cdot) = \tau(e\cdot)/\tau(e)$ and $\phi_f = \tau(f\cdot)/\tau(f)$. It is easy to see that $d(e,f) \leq W_1(\phi_e, \phi_f)$. Then triangle inequality and (3.11) yield $d(e,f) \leq \sqrt{2c\,\mathrm{Ent}(e)} + \sqrt{2c\,\mathrm{Ent}(f)}$. By spectral decomposition theorem of the identity, $\mathrm{Ent}(e) = \int_0^1 \frac{1_A}{\tau(e)} \ln \frac{1_A}{\tau(e)} d\mu$ where A is a Borel set such that $1_A(Id) = e$. Hence we find $\mathrm{Ent}(e) = -\ln \tau(e)$, which gives the first assertion. The equivalent formulation is a simple calculation. □

To conclude this section, we remark that the best possible α in $\Gamma_2 \geq \alpha\Gamma$ sometimes characterizes the dynamical system $(\mathcal{M}, T_t)$. See the example of hyperfinite II_1 factor below.

4. Application to the group von Neumann algebras

Starting from this section, we will investigate a variety of examples which satisfy the assumptions of Theorem 3.4. All the Poincaré type, deviation and transportation inequalities we derived previously will hold in these examples. The key point is to check $\Gamma_2 \geq \alpha\Gamma$.

It may be of independent interest because it means a strictly positive Ricci curvature from the geometric point of view. We consider the Γ_2-criterion for group von Neumann algebras in this section.

Let G be a countable discrete group. In this paper we say that $\psi : \psi \to \mathbb{R}_+$ is a conditional negative definite length (cn-length) function if it vanishes at the identity e, $\psi(g) = \psi(g^{-1})$ and is conditionally negative which means that $\sum_g \xi_g = 0$ implies $\sum_{g,h} \bar{\xi}_g \xi_h \psi(g^{-1}h) \leq 0$. By Schoenberg's theorem, any cn-length function determines an affine representation $(H_\psi, \lambda_\psi, b_\psi)$ and vice versa. Here $\lambda : G \to O(H_\psi)$ is an orthogonal representation over the real Hilbert space H_ψ together with a mapping: $b : G \to H_\psi$ with the cocycle law $b(gh) = b(g) + \lambda_g(b(h))$.

Let $\lambda : G \to B(\ell_2(G))$ be the left regular representation given by $\lambda(g)\delta_h = \delta_{gh}$ where δ_g's form the unit vector basis of $\ell_2(G)$. Let $\mathcal{L}(G) = \lambda(G)''$, the von Neumann algebra generated by $\{\lambda(g) : g \in G\}$. Any $f \in \mathcal{L}(G)$ can be written as

$$f \;=\; \sum_{g \in G} \hat{f}(g)\lambda(g)\;.$$

It is well known that $\tau(f) = \langle \delta_e, f\delta_e \rangle$ defines a faithful normal tracial state and $\tau(f) = \hat{f}(e)$ where e is the identity element. In what follows, we define the semigroup associated to the cn-length function ψ by $T_t(\lambda(g)) = T_t^\psi(\lambda(g)) = \phi_t(g)\lambda(g)$ for $g \in G$, where $\phi_t(g) = e^{-t\psi(g)}$. The infinitesimal generator of T_t is given by $A\lambda(g) = \psi(g)\lambda(g)$. Recall that the Gromov form is defined as

$$K(g,h) \;=\; K_{g,h} \;=\; \frac{1}{2}(\psi(g) + \psi(h) - \psi(g^{-1}h)), \quad g, h \in G\;.$$

Given $f = \sum_{x \in G} \hat{f}(x)\lambda(x)$ and $g = \sum_{y \in G} \hat{g}(y)\lambda(y)$, a straightforward calculation gives

$$\Gamma(f,g) \;=\; \sum_{x,y \in G} \bar{\hat{f}}(x)\hat{g}(y)K(x,y)\lambda(x^{-1}y)\;,$$

$$\Gamma_2(f,g) \;=\; \sum_{x,y \in G} \bar{\hat{f}}(x)\hat{g}(y)K(x,y)^2\lambda(x^{-1}y)\;.$$

Let $\mathcal{A}$ be the subalgebra of $\mathcal{L}(G)$ which consists of elements that can be written as finite combination of $\lambda_g, g \in G$. Then $\mathcal{A}$ is weakly dense in $\mathcal{L}(G)$ such that $A\mathcal{A} \subset \mathcal{A}$ and $T_t\mathcal{A} \subset \mathcal{A}$.

Lemma 4.1. *$\mathcal{A}$ is dense in* $\mathrm{Dom}(A^{1/2})$ *in the graph norm of $A^{1/2}$ and T_t is a standard nc-diffusion semigroup acting on $\mathcal{L}(G)$.*

Proof. For $f = \sum_{g_i \in G} \hat{f}(g_i)\lambda(g_i) \in \mathrm{Dom}(A^{1/2})$, put $f_n = \sum_{i=1}^n \hat{f}(g_i)\lambda(g_i)$. Note that $f \in L_2(\mathcal{L}(G))$. Then

$$\|f_n - f\|_{L_2(\mathcal{L}(G))} \;=\; \tau((f_n^* - f^*)(f_n - f)) \;=\; \sum_{i=n+1}^{\infty} |\hat{f}(g_i)|^2 \to 0, \text{ as } n \to \infty\,.$$

Since $\langle Af, f\rangle = \sum_{i=1}^{\infty} \psi(g_i)|\hat{f}(g_i)|^2 < \infty$, we have

$$\langle A(f_n - f), f_n - f\rangle_{L_2(\mathcal{L}(G),\tau)} \;=\; \sum_{i=n+1}^{\infty} \psi(g_i)|\hat{f}(g_i)|^2 \to 0, \text{ as } n \to \infty\,.$$

Therefore $\mathcal{A}$ is dense in the graph norm. Schoenberg's theorem implies that T_t is completely positive. It can be directly checked that T_t is normal and unital. Since $\psi(g) = \psi(g^{-1})$,

$$\tau(T_t(x)y) \;=\; \langle \delta_e, \sum_g e^{-t\psi(g)}\hat{x}(g)\lambda_g \sum_h \hat{y}(h)\lambda_h\delta_e\rangle \;=\; \sum_g e^{-t\psi(g)}\hat{x}(g)\hat{y}(g^{-1}) \;=\; \tau(xT_ty)\,.$$

Hence T_t is self-adjoint. To check that (T_t) is weak* continuous on $\mathcal{L}(G)$, it suffices to verify that (T_t) is a strongly continuous semigroup on $L_2(\mathcal{L}(G))$. For $f \in L_2(\mathcal{L}(G))$, we have

$$\|T_tf - f\|^2_{L_2(\mathcal{L}(G))} \;=\; \sum_{g\in G}(e^{-t\psi(g)} - 1)^2|\hat{f}(g)|^2 \to 0 \text{ as } t \to 0\,.$$

We have proved that (T_t) is a standard semigroup. Let $f \in \mathcal{A}$. Then

$$\|\Gamma(f,f)\|_1 \;=\; \langle \delta_e, \sum_{g\in G} |\hat{f}(g)|^2 K_{g,g}\delta_e\rangle \;=\; \sum_{g\in G} \psi(g)|\hat{f}(g)|^2 \;=\; \|A^{1/2}f\|_2 < \infty\,.$$

Since $\mathcal{A}$ is dense in $\mathrm{Dom}(A^{1/2})$ in the graph norm, by a similar approximation argument to Lemma 3.8, the above equality holds for all $f \in \mathrm{Dom}(A^{1/2})$ and thus (T_t) is a nc-diffusion semigroup. □

By virtue of Lemma 4.1 and Corollary 3.8, our Poincaré inequalities will follow if the Γ_2-criterion holds. Put $\mathrm{Fix} = \{f \in \mathcal{L}(G) : \psi(f) = 0\}$.

Corollary 4.2. *Let $2 \le p < \infty$ and assume $\Gamma_2(f,f) \ge \alpha\Gamma(f,f)$ for $f \in \mathcal{A}$. Then there exists a constant C such that for all self-adjoint $f \in L_p(\mathcal{L}(G),\tau)$,*

$$\|f - E_{\mathrm{Fix}}(f)\|_p \;\le\; C\alpha^{-1/2}\min\{\sqrt{p}\|\Gamma(f,f)^{1/2}\|_\infty,\; p\|\Gamma(f,f)^{1/2}\|_p\}\,.$$

If $\psi(g) = 0$ only if g is the identity element, then $E_{\mathrm{Fix}}(f) = \tau(f)$ for $f \in \mathcal{L}(G)$.

Among the examples we will consider below, the free group on n generators $\mathbb{F}_n$ satisfies $E_{\mathrm{Fix}}(f) = \tau(f)$ but the finite cyclic group $\mathbb{Z}_n$ has nontrivial Fix. With the help of Lemma

4.1 and Corollary 3.8, we only need to check the Γ_2-criterion on the finitely supported elements in order to fulfill the hypotheses of our main theorem. We call

$$[\Gamma_2(x_i,x_j)] \geq \alpha[\Gamma(x_i,x_j)] \text{ for any } n\in\mathbb{N} \text{ and } x_1,\cdots,x_n\in\mathcal{A}$$

the algebraic Γ_2-condition or (Γ_2-criterion) and abbreviate it to "$\Gamma_2 \geq \alpha\Gamma$ in $\mathcal{L}(G)$". This is the theme of two sections from now on. The following technical lemmas will be used repeatedly.

Lemma 4.3. *Suppose $K=(K_{x,y})_{x,y\in G}$ is a matrix indexed by G with entries in $\mathbb{C}$ and define a sesquilinear form $\Theta:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$, $\Theta(f,g)=\sum_{x,y\in G}\bar{\hat{f}}(x)\hat{g}(y)K_{x,y}\lambda(x^{-1}y)$. Then K is nonnegative definite if and only if Θ is positive.*

Proof. Our proof is based on Lance [32, Proposition 2.1]. Assume K is nonnegative definite. Write $K=X^*X$ for $X=(x_{g,h}), x_{g,h}\in\mathbb{C}$. Then $K=\sum_l(\sum_{g,h}x^*_{lg}x_{lh}\otimes e_{g,h})$. Given $f^1,\cdots,f^n\in\mathcal{A}$, we have

$$\begin{aligned}\Theta(f^i,f^j) &= \sum_{l\in G}\sum_{g,h\in G}\bar{\hat{f}}^i(g)\hat{f}^j(h)x^*_{lg}x_{lh}\lambda(g^{-1}h)\\ &= \sum_{l\in G}\Big(\sum_g \hat{f}^i(g)x_{lg}\lambda_g\Big)^*\Big(\sum_h \hat{f}^j(h)x_{lh}\lambda_h\Big).\end{aligned}$$

Here we understand all indices are finite. Put $\theta^i_l=\sum_g\hat{f}^i(g)x_{lg}\lambda_g$. We then have

$$\Theta = \sum_{l\in G}\sum_{i,j=1}^n \theta_l^{i*}\theta_l^j\otimes e_{i,j} = \sum_{l\in G}\Big(\sum_{i=1}^n\theta^i_l\otimes e_{1,i}\Big)^*\Big(\sum_{j=1}^n\theta^j_l\otimes e_{1,j}\Big),$$

which is positive in $M_n(\mathcal{L}(G))$. Conversely, let $(x_i)=\sum_i x_i\delta_{k_i}\in\ell_2(G)$ and write $g_i={k_i}^{-1}\in G$. Then $(x_i)=\sum_{i\in\mathbb{N}}x_i\delta_{g_i^{-1}}$. Let $f^i=\lambda(k_i)$ so that $(\Theta(f^i,f^j))=(K_{i,j}\lambda(k_i^{-1}k_j))$ is positive in $M_n(\mathcal{L}(G))$ for all $n\in\mathbb{N}$. Then for $h_i=x_i\delta_{g_i}\in\ell_2(G)$, we have for all $n\in\mathbb{N}$

$$\begin{aligned}0 &\leq \langle[\Theta(f^i,f^j)](h_1,\cdots,h_n),(h_1,\cdots,h_n)\rangle_{\ell_2^n(\ell_2(G))}\\ &= \sum_{i,j=1}^n\langle K_{i,j}\lambda(k_i^{-1}k_j)x_j\delta_{g_i},x_i\delta_{g_i}\rangle_{\ell_2(G)} = \sum_{i,j=1}^n K_{i,j}x_j\bar{x}_i,\end{aligned}$$

which implies that K is nonnegative definite. □

The next lemma is useful when we deal with the product of groups. Note that

$$\mathcal{L}(\prod_{i=1}^m G_i)\cong\bar{\otimes}_{i=1}^m\mathcal{L}(G_i).$$

The identification is given by $\lambda(g_1,\cdots,g_m) \mapsto \lambda(g_1)\otimes\cdots\otimes\lambda(g_m)$ for $g_i \in G_i$. We associated the form Γ_2 to a matrix K as follows: $\Gamma_2^K(f,g) = \sum_{x,y\in G} \bar{\hat{f}}(x)\hat{g}(y)K_{x,y}^2\lambda(x^{-1}y)$. In what follows the matrix K will be the Gromov form. If $K = K_1\otimes K_2$, then it is easy to check $\Gamma^K(f^i\otimes g^i, f^j\otimes g^j) = \Gamma^{K_1}(f^i,f^j)\otimes\Gamma^{K_2}(g^i,g^j)$ for $f_i\in\mathcal{L}(G_1), g_i\in\mathcal{L}(G_2)$.

Lemma 4.4. *Let $(K_i)_{i=1}^m$ be nonnegative definite matrices and Γ^{K_i} the associated gradient forms in the sense of Lemma 4.3. Suppose $\Gamma_2^{K_i}\ge\alpha\Gamma^{K_i}$. Then*

$$\Gamma_2^K \ge \alpha\Gamma^K,$$

where $K=\sum_{i=1}^m \mathbb{1}\otimes\cdots\otimes K_i\otimes\cdots\otimes\mathbb{1}$ with K_i in the ith position and in what follows $\mathbb{1}$ always denotes the matrix with every entry equal to 1.

Proof. In light of Lemma 4.3 it suffices to verify $K\bullet K\ge\alpha K$. Here and in the following $A\bullet B$ denotes the Schur product of matrix. Note that trivially $\mathbb{1}\ge 0$. Since $K_i\ge 0$, all the "cross terms" of the form

$$\mathbb{1}\otimes\cdots\otimes K_{i_1}\otimes\cdots\otimes K_{i_2}\otimes\cdots\otimes\mathbb{1}$$

are nonnegative matrices for all $1\le i_1<i_2\le m$. It follows that

$$K\bullet K \;\ge\; \sum_{i=1}^m \mathbb{1}\otimes\cdots\otimes(K_i\bullet K_i)\otimes\cdots\otimes\mathbb{1} \;\ge\; \alpha K\,. \qquad \square$$

4.1. **The free groups.** Let $\mathbb{F}_n$ denote the free group on n generators with length function $\psi=|\cdot|$, where for $g\in\mathbb{F}_n$, $|g|$ is the length of (the freely reduced form of) g. Note that the Gromov form $K(g,h)=|\min(g,h)| := \max\{|w| : g=wg', h=wh'\}$ where $\min(g,h)$ is the longest common prefix subword of g and h.

Proposition 4.5. $\Gamma_2\ge\Gamma$ *holds in* $\mathcal{L}(\mathbb{F}_n)$.

Proof. For a freely reduced word $x\in\mathbb{F}_n$, write $x_i\prec x$ for the prefix subword of x with length i. Following Haagerup's construction, we define a map

$$V:\mathbb{F}_n\to\ell_2(\mathbb{F}_n),\quad x\mapsto V(x)=\sum_{x_i\prec x}\sqrt{2(i-1)}\delta_{x_i}\,.$$

Then we have

$$\widetilde{K}_{x,y} := K_{x,y}^2 - K_{x,y} \;=\; \langle V(x),V(y)\rangle_{\ell_2(\mathbb{F}_n)} \;=\; V(x)^*V(h)\,,$$

where $V(x)^*$ is a row vector and $V(y)$ a column vector. It follows that $\widetilde{K}=(\widetilde{K}_{g,h})_{g,h}$ is a nonnegative definite matrix. We deduce from Lemma 4.3 that $\Gamma_2\ge\Gamma$. $\square$

Remark 4.6. (1) It was shown in [24, Remark 1.3.2] that

$$\|T_t : L_1^0(\mathcal{L}(\mathbb{F}_n)) \to L_\infty(\mathcal{L}(\mathbb{F}_n))\| \le Ct^{-3}.$$

Therefore, Theorem 3.18 and Corollary 3.22 give two different ways to prove the transportation inequality (3.11) for $\mathcal{L}(\mathbb{F}_n)$.

(2) We can construct a sequence of positive definite matrices $(K^{(m)})_m$ such that

$$\big([K_{g,h}^{(m)}]^2\big)_{g,h\in G} \ge \big(K_{g,h}^{(m)}\big)_{g,h\in G}.$$

Indeed, put $K^{(1)} = (K_{g,h})$, and inductively

$$K^{(m+1)} \;=\; K^{(m)} \bullet K^{(m)} - K^{(m)} \;=\; \big([K_{g,h}^{(m)}]^2 - K_{g,h}^{(m)}\big), \quad m \in \mathbb{N}.$$

Suppose $K_{g,h}^{(1)} = k$. Then $K_{g,h}^{(m)} = f(k)$ for some polynomial f. By induction, $(f(k))$ is an increasing sequence. Using the elementary summation formula

$$\sum_{i=1}^{m} i^p = \frac{(m+1)^{p+1}}{p+1} + \sum_{j=1}^{p} \frac{B_j}{p-j+1}\binom{p}{j}(m+1)^{p-j+1}$$

where B_j denotes a Bernoulli number, we can find a sequence a_i such that $f(k) = \sum_{i=1}^{k} a_i$ for all $k \in \mathbb{N}$. a_i is of the form $a_i = \sum_{p=0}^{2^{m-1}-1} \lambda_p i^p$ for $\lambda_p \in \mathbb{R}$ and $a_i = f(i+1) - f(i) \ge 0$. Then we define a map

$$W : \mathbb{F}_n \to \ell_2(\mathbb{F}_n), \quad g \mapsto W(g) = \sum_{g_i \prec g} \sqrt{a_i}\,\delta_{g_i}\,.$$

By construction, we have

$$K_{g,h}^{(m)} \;=\; \langle W(g), W(h)\rangle_{\ell_2(\mathbb{F}_n)} \;=\; W(g)^* W(h),$$

which implies the positivity of $K^{(m)}$. The point here is that $\mathcal{L}(\mathbb{F}_n)$ admits infinitely many semigroups with positive "curvature".

The particular case $n = 1$ gives some interesting results in classical Fourier analysis. Indeed, $\mathcal{L}(\mathbb{F}_1) = \mathcal{L}(\mathbb{Z}) = L_\infty(\mathbb{T})$ and $L_p(\mathcal{L}(\mathbb{F}_1)) = L_p(\mathbb{T})$ after identifying $\lambda(k)(x) = e^{2\pi i k x}$. In this case

$$K(j,k) = \begin{cases} \min(|j|,|k|), & jk > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 4.7. *Let $2 \le p < \infty$. Then there exists constants C and C' such that for all $f \in L_p(\mathbb{T})$, we have*

$$\|f - \hat{f}(0)\|_p \;\le\; C\sqrt{p}\,\Bigg\| \sum_{j,k\in\mathbb{Z}, jk>0} \bar{\hat{f}}(j)\hat{f}(k)\min(|j|,|k|)e^{2\pi i(k-j)\cdot} \Bigg\|_\infty^{1/2},$$

$$\|f-\hat{f}(0)\|_p \le C'p \left\| \sum_{j,k\in\mathbb{Z},jk>0} \bar{\hat{f}}(j)\hat{f}(k)\min(|j|,|k|)e^{2\pi i(k-j)\cdot} \right\|_{p/2}^{1/2}.$$

Remark 4.8. Observe that this example is purely commutative. However, commutative probability theory seems insufficient to establish these inequalities. Intuitively, the multiplier $|j|$ corresponds to $\Delta^{1/2}$. The Markov process generated by $\Delta^{1/2}$ is the Cauchy process with discontinuous path. The classical diffusion theory does not apply here. But it is still nc-diffusion so that our noncommutative theory is essential in this regard. In general, whenever the process has discontinuous path but its semigroup still satisfies our assumptions, the noncommutative theory seems to be a natural choice due to the existence of Markov dilation with a.u. continuous path as stated in Theorem 2.2. We will have more examples of this kind in the following.

4.2. **Application to the noncommutative tori $\mathcal{R}_\Theta$.** We recall the definition following [23]. Let Θ be a $d\times d$ antisymmetric matrix with entries $0\le\theta_{ij}<1$. The noncommutative torus (or the rotation algebra) with d generators associated to Θ is the von Neumann algebra $\mathcal{R}_\Theta$ generated by d unitaries $u_1,\cdots,u_d$ satisfying $u_ju_k=e^{2\pi i\theta_{jk}}u_ku_j$. Every element of $\mathcal{R}_\Theta$ is in the closure of the span of words of the form $w_k=u_1^{k_1}\cdots u_d^{k_d}$ for $k=(k_1,\cdots,k_d)\in\mathbb{Z}^d$. $\mathcal{R}_\Theta$ admits a unique normal faithful trace τ given by $\tau(x)=\hat{x}(0)$ where $x=\sum_{k\in\mathbb{Z}^d}\hat{x}(k)u_1^{k_1}\cdots u_d^{k_d}\in\mathbb{R}_\Theta$. Our goal is to show that $\mathcal{R}_\Theta$ admits a standard nc-diffusion semigroup with the Γ_2-criterion. We start with the von Neumann algebra of $\mathbb{Z}^d$. It is well known that $\mathcal{L}(\mathbb{Z}^d)\cong L_\infty(\mathbb{T}^d)$. We define $\psi(k)=\|k\|_1=\sum_{i=1}^d|k_i|$ for $k=(k_1,\cdots,k_d)\in\mathbb{Z}^d$. Clearly, ψ is a cn-length function and thus generate a standard nc-diffusion semigroup P_t by Lemma 4.1. In fact $P_t=\widetilde{P}_t^{\otimes d}$ where $\widetilde{P}_t$ is the Poisson semigroup on $L_\infty(\mathbb{T})$.

Proposition 4.9. *Let Γ be the gradient form associated to P_t. Then $\Gamma_2\ge\Gamma$ in $\mathcal{L}(\mathbb{Z}^d)$.*

Proof. Let K^d be the Gromov form associated with ψ. A calculation shows that $K^d(j,k)=K(j_1,k_1)+\cdots+K(j_d,k_d)$ for $j=(j_1,\cdots,j_d),k=(k_1,\cdots,k_d)\in\mathbb{Z}^d$, where K is the Gromov form of $\mathbb{Z}=\mathbb{F}_1$ considered in the proceeding subsection. Alternatively, we may write $K^d=\sum_{i=1}^d \mathbb{1}\otimes\cdots\otimes K\otimes\cdots\otimes\mathbb{1}$ where K is in the ith position. But we know from Proposition 4.5 that $\Gamma_2\ge\Gamma$ in $\mathcal{L}(\mathbb{Z})$. The assertion follows from Lemma 4.3. $\square$

Proposition 4.10. *$\mathcal{R}_\Theta$ admits a standard nc-diffusion semigroup with $\Gamma_2\ge\Gamma$.*

Proof. Let $k\in\mathbb{Z}^d$. Consider an action $\alpha:\mathbb{T}^d\to\mathrm{Aut}(\mathcal{R}_\Theta)$ given by for $s\in\mathbb{T}^d$, $\alpha_s(u_1^{k_1}\cdots u_d^{k_d})=e^{2\pi i\sum_{j=1}^d k_js_j}u_1^{k_1}\cdots u_d^{k_d}$. It is easy to check that α_s is a trace preserving

automorphism. Define a map

$$\pi : \mathcal{R}_\Theta \to L_\infty(\mathbb{T}^d)\bar{\otimes}\mathcal{R}_\Theta, \quad w_k = u_1^{k_1}\cdots u_d^{k_d} \mapsto \pi(w_k)(s) = \alpha_s(w_k) = e^{2\pi i\langle k,s\rangle}u_1^{k_1}\cdots u_d^{k_d} \, .$$

Then π is an injective $*$-homomorphism. Define $T_t : \mathcal{R}_\Theta \to \mathcal{R}_\Theta, T_t(w_k) = e^{-t\|k\|_1}w_k$. We claim that $(T_t)_{t\geq 0}$ is the desired semigroup. Indeed, by Lemma 5.1 and Proposition 4.9, $P_t \otimes Id$ acting on $L_\infty(\mathbb{T}^d)\bar{\otimes}\mathcal{R}_\Theta$ is a standard nc-diffusion semigroup and satisfies $\Gamma_2 \geq \Gamma$. Then since π is injective and

$$P_t \otimes Id(\pi(w_k)) \;=\; e^{-t\|k\|_1}e^{2\pi i\langle k,\cdot\rangle} \otimes u_1^{k_1}\cdots u_d^{k_d} \;=\; \pi(T_t(w_k)) \, ,$$

we deduce that T_t is a standard nc-diffusion semigroup with the Γ_2-criterion. □

4.3. **The finite cyclic group $\mathbb{Z}_n$.** We consider the group von Neumann algebra $\mathcal{L}(\mathbb{Z}_n)$ in this subsection. Let $(e_j)_{j=1}^n$ be the standard basis of $\mathbb{C}^n$. Each e_j can be regarded as a vector in $\mathbb{R}^{2n}$ by canonical identification. Given $k \in \mathbb{Z}_n$, define the $2n \times 2n$ diagonal matrix $\alpha_k = (e^{2\pi ikj/n})_{j=0}^{n-1}$ where each $e^{2\pi ikj/n}$ is on diagonal and is identified with the 2×2 rotation matrix

$$\begin{pmatrix} \cos(2\pi kj/n) & -\sin(2\pi kj/n) \\ \sin(2\pi kj/n) & \cos(2\pi kj/n) \end{pmatrix}.$$

Consider the finite cyclic group $\mathbb{Z}_n$ with 1-cocycle structure $(b,\alpha,\mathbb{R}^{2n})$, where

$$b(k) \;=\; \frac{1}{\sqrt{n}}\Big(\sum_{j=1}^n \alpha_k(e_j) - e_j\Big) \;=\; \frac{1}{\sqrt{n}}\sum_{j=1}^n \begin{pmatrix}\cos(2\pi k(j-1)/n) - 1 \\ \sin(2\pi k(j-1)/n)\end{pmatrix} \otimes e_j \, .$$

Lemma 4.11. *Let $K(k,h)$ be the Gromov form. Then $K(k,h) = \langle b(k), b(h)\rangle$.*

Proof. Since the length function $\psi(k) = \|b(k)\|^2$, by the cocycle property,

$$\begin{aligned} K(k,h) \;&=\; \frac{1}{2}(\|b(k)\|^2 + \|b(h)\|^2 - \|b(h-k)\|^2) \\ &=\; \frac{1}{2}(\|b(-k)\|^2 + \|\alpha_{-k}(b(h))\|^2 - \|\alpha_{-k}(b(h)) + b(-k)\|^2) \\ &=\; -\langle b(-k), \alpha_{-k}(b(j))\rangle = -\langle \alpha_k(b(-k)), b(h)\rangle \\ &=\; \langle b(k), b(h)\rangle \, . \end{aligned}$$

□

Clearly, $K(k,h) = 0$ if $k = 0$ or $h = 0$. For $k, h \neq 0$, a computation gives

$$\begin{aligned} K(k,h) \;&=\; \frac{1}{n}\sum_{j=0}^{n-1}\big[(1-\cos(2\pi kj/n))(1-\cos(2\pi hj/n)) + \sin(2\pi kj/n)\sin(2\pi hj/n)\big] \\ &=\; 1 + \frac{1}{n}\sum_{j=0}^{n-1}\cos\Big(\frac{2\pi(k-h)j}{n}\Big) = 1 + \delta_{k,h} \, , \end{aligned}$$

where $\delta_{k,h}$ is the Kronecker delta function. It follows that $\psi(k) = 2(1-\delta_{k,0})$. For reasons that will become clear later, we normalize ψ and still denote it by ψ so that $\psi(k) = 1-\delta_{k,0}$ for $k \in \mathbb{Z}_n$. Then the associated Gromov form satisfies $K_{k,h} = \frac{1}{2}(1+\delta_{k,h})$ for $k, h \neq 0$ and $(K_{k,h}^2 - \frac{1}{2}K_{k,h}) \geq 0$. It is an immediate consequence of Lemma 4.3 that $\Gamma_2 \geq \frac{1}{2}\Gamma$ in $\mathcal{L}(\mathbb{Z}_n)$. In fact, we can do better.

Proposition 4.12. *For all $0 < \alpha \leq \frac{n+2}{2n}$, we have $\Gamma_2 \geq \alpha\Gamma$ in $\mathcal{L}(\mathbb{Z}_n)$. Moreover, $\alpha_n = \frac{n+2}{2n}$ is the largest possible α with the Γ_2-criterion.*

Proof. Note that the $n \times n$ matrix K can be written as a block matrix

$$K = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}(I_{n-1} + \mathbb{1}_{n-1}) \end{pmatrix}$$

where I_{n-1} is the $n-1$ dimensional identity matrix and every entry of $\mathbb{1}_{n-1}$ is 1. Write $\widehat{K} = \frac{1}{2}(I_{n-1} + \mathbb{1}_{n-1})$. Since $\mathbb{1}_{n-1} \leq (n-1)I_{n-1}$, for $0 < \alpha \leq \frac{n+2}{2n}$ we have

$$\begin{aligned} 4\widehat{K} \bullet \widehat{K} - 4\alpha\widehat{K} &= (3-2\alpha)I_{n-1} - (2\alpha-1)\mathbb{1}_{n-1} \\ &\geq (2+n-2\alpha n)I_{n-1} \geq 0 . \end{aligned}$$

Plugging $\boldsymbol{x} = (\frac{1}{\sqrt{n-1}}, \cdots, \frac{1}{\sqrt{n-1}})$ into $\boldsymbol{x}'(\widehat{K} \bullet \widehat{K} - \alpha\widehat{K})\boldsymbol{x} \geq 0$ reveals that $\alpha_n = \frac{n+2}{2n}$ is sharp. Then Lemma 4.3 leads to the Γ_2-criterion. □

4.4. **The discrete Heisenberg group $H_3(\mathbb{Z}_n)$.** Let $H_3(\mathbb{Z}_n) = \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$ be the discrete Heisenberg group over $\mathbb{Z}_n$. We will write H for $H_3(\mathbb{Z}_n)$ as long as there is no confusion. The multiplication in H is given by

$$(a,b,c)(a',b',c') = (a+a'+bc', b+b', c+c'), \quad (a,b,c),(a',b',c') \in H .$$

Proposition 4.13. *Let $\psi(a,b,c) = 2 - \delta_{b,0} - \delta_{c,0}$. Then*

(1) *ψ is conditional negative and thus the semigroup (T_t) determined by ψ is a standard nc-diffusion semigroup.*
(2) *Let Γ be the gradient form associated to ψ. Then $\Gamma_2 \geq \frac{n+2}{2n}\Gamma$ in $\mathcal{L}(H)$.*

Proof. (1) The length function of $\mathbb{Z}_n$ considered in Subsection 4.3 is given by $\widetilde{\psi}(k) = (1-\delta_{k,0})$, which extends to $\mathbb{Z}_n \times \mathbb{Z}_n$ as $\widetilde{\psi}(k,l) = (1-\delta_{k,0}) + (1-\delta_{l,0})$. Define a group homomorphism

$$\beta : H \to \mathbb{Z}_n \times \mathbb{Z}_n, \quad (a,b,c) \mapsto (b,c).$$

Since $\widetilde{\psi}$ is conditional negative, it follows from the definition that $\psi = \widetilde{\psi} \circ \beta$ is also conditional negative. Lemma 4.1 yields that (T_t) is a standard nc-diffusion semigroup.

(2) Let K and $\widetilde{K}$ be the Gromov form of (H,ψ) and $(\mathbb{Z}_n,\widetilde{\psi})$ respectively. A calculation shows that for indices $(a,b,c),(a',b',c')\in H$,

$$\begin{aligned} K((a,b,c),(a',b',c')) &= \widetilde{K}(b,b')+\widetilde{K}(c,c') \\ &= (\widetilde{K}\otimes \mathbb{1}+\mathbb{1}\otimes \widetilde{K})((b,b')\otimes(c,c'))\,. \end{aligned}$$

By Proposition 4.12 and Lemma 4.4 with $m=2$, we have $\Gamma_2^K \geq \frac{n+2}{2n}\Gamma^K$ in $\mathcal{L}(H)$, as desired. □

Let $e_{i,j}$ be the standard basis of the matrix algebra $M_n(\mathbb{C})$ and δ_j the standard basis of $\ell_2(\mathbb{Z}_n)$. Define the diagonal matrix $u_k=\sum_{j=1}^n e^{2\pi ik(j-1)/n}\otimes e_{j,j}$ and the shift operator $v_l(\delta_j)=\delta_{j+l}$ which is nothing but the left regular representation of $\mathbb{Z}_n$ on $\ell_2(\mathbb{Z}_n)$. It is easy to see that $u_k,v_l\in M_n=B(\ell_2(\mathbb{Z}_n))$ and they satisfy $u_kv_l=e^{2\pi ikl/n}v_lu_k$.

Proposition 4.14. *Let $\mathcal{L}(H)$ be the group von Neumann algebra of H. Then*

$$\mathcal{L}(H)\cong L_\infty(\mathbb{Z}_n^2)\oplus M_n\oplus \mathcal{M}_2\oplus\cdots\oplus\mathcal{M}_{n-1}\,,$$

where $\mathcal{M}_x, x=2,\cdots,n-1$ are von Neumann algebras acting on $\ell_2(\mathbb{Z}_n^2)$. Moreover, if T_t is the semigroup associated to $\psi(a,b,c)=2-\delta_{b,0}-\delta_{c,0}$, then T_t leaves each component invariant and $T_t|_{\mathcal{M}_x}$is a standard nc-diffusion semigroup.

Proof. Let us first determine the center of $\mathcal{L}(H)$ denoted by $\mathcal{Z}$. The identity

$$\lambda(a,b,c)\lambda(a',b',c')=\lambda(a',b',c')\lambda(a,b,c)$$

for all $(a',b',c')\in H$ holds if and only if $b=c=0$. Thus $\mathcal{L}(\mathbb{Z}_n,0,0)\subset\mathcal{Z}$. Let $\mathcal{F}$ denote the discrete Fourier transform of the first component on $\ell_2(H)$. For $\delta_{(x,0,0)}\in\ell_2(\mathbb{Z}_n,0,0)$, we have

$$\mathcal{F}(\delta_{(x,0,0)}) \;=\; \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} e^{-\frac{2\pi ikx}{n}}\delta_{(k,0,0)}\,.$$

A calculation gives

$$\mathcal{F}\lambda(a,0,0)\mathcal{F}^{-1}\delta_{(x,0,0)} \;=\; e^{-\frac{2\pi iax}{n}}\delta_{(x,0,0)}\,.$$

This shows that $\mathcal{F}\mathcal{L}(\mathbb{Z}_n,0,0)\mathcal{F}^{-1}=\{e^{-\frac{2\pi ia\cdot}{n}}:a\in\mathbb{Z}_n\}''=L_\infty(\mathbb{Z}_n)$. Since for fixed $x\in\mathbb{Z}_n$, $\delta_{(x,\cdot,\cdot)}\in\ell_2(\mathbb{Z}_n^2)$, we have the Hilbert space decomposition $\ell_2(H)=\bigoplus_{x\in\mathbb{Z}_n}\ell_2(\mathbb{Z}_n^2)$. Then by the decomposition theorem of von Neumann algebras for subalgebras of the center, see e.g. [49, Theorem IV.8.21],

$$\mathcal{L}(H)\cong\bigoplus_{x\in\mathbb{Z}_n}\mathcal{M}_x\,,$$

where $\mathcal{M}_x$ is determined by the unitary $\mathcal{F}$, the discrete Fourier transform on the first component. Let $q_x\in\ell_\infty(\mathbb{Z}_n,0,0)$ be the central projection given by $q_x:\ell_2(H)\to\operatorname{span}\{\delta_{(x,y,z)}: y,z\in\mathbb{Z}_n\}$. Put $p_x=\mathcal{F}^{-1}q_x\mathcal{F}\in\mathcal{L}(\mathbb{Z}_n,0,0)$. Then p_x is a central projection, $\sum_{x=0}^{n-1}p_x=1$

and $p_x\mathcal{L}(H) = \mathcal{F}^{-1}\mathcal{M}_x\mathcal{F}$. We observe that $T_t\lambda(a,0,0) = \lambda(a,0,0)$. Hence $\mathcal{L}(\mathbb{Z}_n,0,0)$ is contained in the multiplicative domain of T_t. Then by the property of multiplicative domain (see e.g. [40, Theorem 3.18]) for all $x \in \mathbb{Z}_n, \xi \in \mathcal{L}(H)$, $T_t(p_x\xi) = p_xT_t(\xi) \in p_x\mathcal{L}(H)$. Let $\eta \in \mathcal{M}_x$ with $p_x\xi = \mathcal{F}^{-1}\eta\mathcal{F}$. We define $\widetilde{T}_t\eta = \mathcal{F}T_t(p_x\xi)\mathcal{F}^{-1}$. Since $\mathcal{F}$ is a unitary, $\widetilde{T}_t$ restricted to $\mathcal{M}_x$ is a standard nc-diffusion semigroup. By abuse of notation, we will write T_t for $\widetilde{T}_t$ on $\mathcal{M}_x$ in the future.

To get more precise description of $\mathcal{M}_x$, we define a family of maps for $x \in \mathbb{Z}_n$

$$\begin{aligned} &\pi_x : \mathcal{L}(H) \to B(\ell_2(\mathbb{Z}_n^2)) \\ &\lambda(a,b,c) \mapsto \pi_x(\lambda(a,b,c))\delta_{(x,k,l)} \;=\; \mathcal{F}\lambda(a,b,c)\mathcal{F}^{-1}\delta_{(x,k,l)} \;=\; e^{-\frac{2\pi i x(a+bl)}{n}}\lambda_{(b,c)}\delta_{(k,l)}\ , \end{aligned}$$

where $\lambda(b,c)$ is the shift operator on $\ell_2(\mathbb{Z}_n^2)$ given by $\lambda(b,c)\delta_{(k,l)} = \delta_{(k+b,l+c)}$. Then

$$\mathcal{M}_x = \{\pi_x(\lambda(a,b,c)) : (a,b,c) \in H\}'' = \{e^{-\frac{2\pi i a x}{n}} v_b \otimes (v_c u_{-xb}) : (a,b,c) \in H\}''\ .$$

Here we have used the convention $\lambda(b,c) = v_b \otimes v_c$. If $x = 0$, we have

$$\mathcal{M}_0 = \{\lambda(b,c) : (b,c) \in \mathbb{Z}_n^2\}'' = \mathcal{L}(\mathbb{Z}_n^2) = L_\infty(\mathbb{Z}_n^2).$$

If $x = 1$, it can be checked that $\{v_c u_{-b} : (b,c) \in \mathbb{Z}_n^2\}'' = M_n$, see e.g. [12, Theorem VII.5.1]. Define for $(b,c) \in \mathbb{Z}_n^2$

$$\rho(v_b \otimes (v_c u_{-b})) \;=\; v_c u_{-b}\ .$$

Then ρ is a $*$-isomorphism and thus $\mathcal{M}_1 = M_n$. $\square$

Consider the semigroup T_t acting on $M_n(\mathbb{C})$ defined by $T_t|_{M_n(\mathbb{C})}$ in the preceding proposition. Explicitly, T_t is determined by $T_t(v_cu_b) = e^{-t\psi(b,c)}(v_cu_b)$ where $\psi(b,c) = 2-\delta_{b,0}-\delta_{c,0}$. Then the Γ_2-criterion for M_n follows from Proposition 4.13. We record this fact below.

Proposition 4.15. *M_n admits a standard nc-diffusion semigroup $(T_t)_{t\ge 0}$. Let Γ^{M_n} be the gradient form associated to T_t. Then $\Gamma_2^{M_n} \ge \frac{n+2}{2n}\Gamma^{M_n}$ in M_n.*

4.5. **Application to the generalized Walsh system.** Let us recall some basic facts about the Walsh system following [15]. Let $\Omega_n^m = \{1, e^{2\pi i/n}, e^{2\pi i 2/n}, \cdots, e^{2\pi i(n-1)/n}\}^m$ be the m-dim discrete cube equipped with uniform probability measure P. Let $\omega_j, j = 1,\cdots,m$ denote the jth coordinate function on Ω_n^m. For a nonempty subset $B \subset \{1,\cdots,m\}$ and $\boldsymbol{x} = (x_1,\cdots,x_m) \in \mathbb{Z}_n^m$, define

$$\omega_B(\boldsymbol{x}) \;=\; \prod_{j\in B}\omega_j^{x_j},$$

and $\omega_\emptyset = 1$. Put $G = \{\omega_B(\boldsymbol{x}) : B \subset \{1,\cdots,m\}, \boldsymbol{x} \in \mathbb{Z}_n^m\}$. Then G is clearly a group and $L_\infty(\Omega_n^m)$ is spanned by the elements of G.

We consider the abelian group $\mathbb{Z}_n^m$. Define

$$\psi(x_1, \cdots, x_m) \;=\; (n - \delta_{x_1} - \cdots - \delta_{x_m}), \quad (x_1, \cdots, x_m) \in \mathbb{Z}_n^m \;,$$

where $\delta_x = \delta_{x,0}$. Given $\boldsymbol{x} = (x_1, \cdots, x_m)$, put $B_{\boldsymbol{x}} = \{i : x_i \neq 0, i = 1, \cdots, m\}$. Clearly $\psi(x_1, \cdots, x_m) = |B_{\boldsymbol{x}}|$, where $|B|$ is the cardinality of B. A similar argument to the proof of Proposition 4.13 shows that ψ is a cn-length function and the associated Γ form satisfies

$$\Gamma_2 \geq \frac{n+2}{2n}\Gamma \text{ in } \mathcal{L}(\mathbb{Z}_n^m) \;. \tag{4.1}$$

Here the constant $\frac{n+2}{2n}$ is given by Proposition 4.12. Define a map

$$\beta : \mathbb{Z}_n^m \to G, \quad \boldsymbol{x} = (x_1, \cdots, x_m) \mapsto \prod_{j \in B_{\boldsymbol{x}}} \omega_j^{x_j} \;.$$

It is easy to check that β is a group isomorphism from $\mathbb{Z}_n^m$ to G. The idea here is to convert addition to multiplication. Under the identification β, $\mathcal{L}(\mathbb{Z}_n^m) = L_\infty(\Omega_n^m)$ and thus every $f \in \mathcal{L}(\mathbb{Z}_n^m)$ can be written as

$$f \;=\; \mathbb{E}(f) + \sum_{\boldsymbol{x} \in \mathbb{Z}_n^m, \boldsymbol{x} \neq \boldsymbol{0}} \hat{f}_{\boldsymbol{x}} \prod_{j \in B_{\boldsymbol{x}}} \omega_j^{x_j} \;,$$

where $\mathbb{E}(f) = \tau(f)$ is the expectation associated to the uniform probability. By abuse of notation, we still denote by ψ the cn-length function induced by β on $\{\omega_B\}$, i.e.

$$\psi(\omega_{B_{\boldsymbol{x}}}) \;=\; \psi(\beta(\boldsymbol{x})) := \psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{Z}_n^m \;.$$

Then we have

$$\psi(\omega_{B_{\boldsymbol{x}}}) \;=\; \psi(x_1, \cdots, x_m) \;=\; |B_{\boldsymbol{x}}| \;.$$

Therefore the infinitesimal generator A of the heat semigroup T_t in this case is the number operator for the generalized Walsh system which counts non-zero elements

$$A\,\omega_B \;=\; |B|\omega_B \;.$$

Moreover, it follows from (4.1) that $\Gamma_2 \geq \frac{n+2}{2n}\Gamma$ in $L_\infty(\Omega_n^m)$. The case $n = 2$ is of particular interest. Indeed, we may write $f \in \mathcal{L}(\mathbb{Z}_2^m)$ as

$$f \;=\; \mathbb{E}(f) + \sum_{B \subset \{1, \cdots, m\}, B \neq \emptyset} \hat{f}_B \omega_B \;.$$

Note that in this case $\omega_B^{-1}\omega_C = \omega_{B \triangle C}$. Then the Gromov form of $\{\omega_B\}$ is given by

$$K(\omega_B, \omega_C) \;=\; \frac{1}{2}(|B| + |C| - |B \triangle C|) \;=\; |B \cap C| \;.$$

Hence, we find the gradient form

$$\Gamma(f, f) \;=\; \sum_{B, C \subset \{1, \cdots, m\}} \bar{\hat{f}}_B \hat{f}_C |B \cap C| \omega_{B \triangle C} \;.$$

Let $e_j = (1, \cdots, -1, \cdots, 1)$ where -1 is only at the jth position. For $x \in \Omega_n^m$, put $(\partial_j f)(x) = \frac{1}{2}(f(x) - f(xe_j))$ and define the discrete gradient $\nabla f = (\partial_j f)_{j=1}^m$. Then a calculation gives $\Gamma(f,f) = |\nabla f|^2$, where $|\cdot|$ is the Euclidean norm of a vector in $\mathbb{C}^n$. If we simply write $|\nabla f| = \Gamma(f,f)^{1/2}$ for any $n = 2, 3, \cdots$, our Poincaré inequalities for the generalized Walsh system is a dimension m free estimate.

Corollary 4.16. *Let $2 \le p < \infty$. Then for all $f \in L_p(\Omega_n^m, P)$,*

$$\|f - \mathbb{E}(f)\|_p \le C\sqrt{\frac{2n}{n+2}} \min\{\sqrt{p}\| \, |\nabla f| \, \|_\infty, \, p\| \, |\nabla f| \, \|_p\} .$$

One of Efraim and Lust-Piquard's main results in [15] asserts that for $2 < p < \infty$ and $f \in L_p(\Omega_2^m, P)$,

$$\|f - \mathbb{E}(f)\|_p \le C\sqrt{p}\| \, |\nabla f| \, \|_p .$$

Our version for the case $n = 2$ is weaker. But our approach works from general n while their result is only for $n = 2$. For arbitrary $n > 2$, it is unclear to us whether it is possible to obtain similar results based on their method. Moreover, their concentration inequality [15, Corollary 4.2] due to Bobkov and Götze is now a special case of our (3.5) with the same order. Efraim and Lust-Piquard's inequality would follow from our general theory if (1.5) were true.

4.6. **The q-Gaussian algebras.** We first recall some definitions and basic facts following [8]. Throughout this section $-1 \le q \le 1$. Let $\mathcal{H}$ be a separable real Hilbert space with complexification $\mathcal{H}_\mathbb{C}$. Let $(F_q(\mathcal{H}), \langle \cdot, \cdot \rangle_q)$ be the q-Fock space with vacuum vector Ω and $\Gamma_q(\mathcal{H})$ the q-Gaussian algebra which is the von Neumann algebra generated by $s(f) = l(f) + l^*(f)$ for $f \in \mathcal{H}$ where

$$l^*(f) f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n$$

and

$$l(f) f_1 \otimes \cdots \otimes f_n = \sum_{j=1}^n q^{j-1} \langle f_j, f \rangle f_1 \otimes \cdots \otimes f_{j-1} \otimes f_{j+1} \otimes \cdots \otimes f_n$$

are the creation and annihilation operators respectively. The vacuum vector gives rise to a canonical tracial state $\tau_q(X) = \langle X\Omega, \Omega \rangle_q$ for $X \in \Gamma_q(\mathcal{H})$. The q-Ornstein-Uhlenbeck semigroup $T_t^q = \Gamma_q(e^{-t} I_\mathcal{H})$ is a standard semigroup and extends to a semigroup of contractions on L_p spaces. The generator on L_2 is the number operator N^q which acts on the Wick product by

$$N^q W(f_1 \otimes \cdots \otimes f)n) = nW(f_1 \otimes \cdots \otimes f_n), \quad f_1, \cdots, f_n \in \mathcal{H}_\mathbb{C} ,$$

where W is the Wick operator. It is easy to check that (T_t) is a nc-diffusion semigroup.

Let ℓ_2^n be the real Hilbert space with dimension n and $\{e_1, \cdots, e_n\}$ an orthonormal basis. For $j = 1, \cdots, n$, consider the embedding

$$\iota_j : \mathcal{H} \to \mathcal{H} \otimes \ell_2^n, \quad h \mapsto h \otimes e_j \, .$$

According to [8, Theorem 2.11], there exists a unique map $\Gamma_q(\iota) : \Gamma_q(\mathcal{H}) \to \Gamma_q(\mathcal{H} \otimes \ell_2^n)$ such that $\Gamma_q(\iota_j)(s(h)) = s(h \otimes e_j)$. The map $\Gamma_q(\iota)$ is linear, bounded, unital completely positive and preserves the canonical trace. Define $s_j^q(h) = s(h \otimes e_j)$. If $q = 1$, we write $g_j(h) = s(h \otimes e_j)$ and it is well-known $g_j(h)$ is a standard Gaussian random variable if $\|h\| = 1$. For $h \in \mathcal{H}$, put

$$u_n(h) \;=\; \frac{1}{\sqrt{n}} \sum_{j=1}^{n} s_j^q(h) \otimes g_j(h) \, .$$

Write $\mathcal{M}_n^q = \Gamma_q(\mathcal{H} \otimes \ell_2^n)$. We consider the von Neumann algebra ultraproduct $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_n^q \bar{\otimes} \mathcal{M}_n^1$. Any element of $\mathcal{M}$ can be written as $u_\omega(h) = (u_n(h))^\bullet$. We need the following fact proved in [2].

Lemma 4.17. *The map $s(h) \mapsto u_\omega(h)$ extends to an injective trace preserving *-homomorphism $\pi : \Gamma_q(\mathcal{H}) \to \mathcal{M}$. Moreover, for $x \in \Gamma_q(\mathcal{H})$,*

$$\pi(T_t^q(x)) \;=\; (Id \otimes T_t^1)^\bullet \pi(x) \, .$$

Proposition 4.18. *For $-1 \le q \le 1$, $\Gamma_2 \ge \Gamma$ in $\Gamma_q(\mathcal{H})$.*

Proof. Since π is injective, it suffices to prove $\pi\Gamma_2^{N^q} \ge \pi\Gamma^{N^q}$ in $\Gamma_q(\mathcal{H})$. By Lemma 4.17, we have $\pi N^q(x) = (Id \otimes N^1)^\bullet \pi(x)$. It follows that

$$\pi(\Gamma^{N^q}(x, y)) \;=\; \Gamma^{(Id \otimes N^1)^\bullet}(\pi(x), \pi(y)),$$

and similar identity is true for $\Gamma_2^{N^q}$. It is proved in [2] by using the central limit theorem of Biane [6] that $\Gamma_2 \ge \Gamma$ in $\Gamma_1(\mathcal{H})$. Here $\Gamma_1(\mathcal{H})$ is the von Neumann algebra acting on the symmetric Fock space. It follows that for all $n \in \mathbb{N}$ and in $\mathcal{M}_n^q \otimes \mathcal{M}_n^1$, $\Gamma_2^{Id \otimes N^1} \;\ge\; \Gamma^{Id \otimes N^1}$. Hence, we find

$$\Gamma_2^{(Id \otimes N^1)^\bullet}(\pi(x_i), \pi(x_j)) \;\ge\; \Gamma^{(Id \otimes N^1)^\bullet}(\pi(x_i), \pi(x_j))$$

where for any $m \in \mathbb{N}$ and $x_i \in \Gamma_q(\mathcal{H}), i = 1, \cdots, m$, as desired. □

4.7. **The hyperfinite II_1 factor.** Our goal in this subsection is to show that the hyperfinite II_1 factor R admits different standard nc-diffusion semigroups with Γ_2-criterion and that the best possible α characterizes the corresponding dynamical system. It is well known that R can be approximated by matrix algebras $\{M_{n^k} : k \in \mathbb{N}\}$. We will embed $M_{n^{m/2}}$ to the group von Neumann algebra of generalized discrete Heisenberg group $H_n^{m+1} = \mathbb{Z}_n/2 \times \mathbb{Z}_n^m$.

Let $\Theta = (\theta_{jk})$ be an antisymmetric $m \times m$ matrix with $\theta_{jk} = \frac{1}{2}$ if $j < k$. The multiplication in $H_n^{m+1} = \mathbb{Z}_n/2 \times \mathbb{Z}_n^m$ is given by

$$(x, \xi)(y, \eta) \ = \ (x + y + B(\xi, \eta), \xi + \eta) \ ,$$

where $B : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n/2$ is a bilinear form given by $B(\xi, \eta) = \sum_{j,k=1}^m \theta_{jk}\xi_j\eta_k = \langle \xi, \Theta\eta\rangle$. For $(r, \xi) \in H_n^{m+1}$, put $\psi(r, \xi) = \sum_{j=1}^m 1 - \delta_{\xi_j,0} = \#\{\xi_j \neq 0\}$. Define a semigroup acting on $\mathcal{L}(H_n^{m+1})$ by $T_t\lambda(r, \xi) = e^{-t\psi}\lambda(r, \xi)$ for $\lambda(r, \xi) \in \mathcal{L}(H_n^{m+1})$ and $t \geq 0$. Using Lemma 4.4, a similar argument to Proposition 4.13 shows that $(T_t)_{t\geq 0}$ is a standard nc-diffusion semigroup and that the associated gradient form satisfies $\Gamma_2 \geq \frac{n+2}{2n}\Gamma$ in $\mathcal{L}(H_n^{m+1})$.

Lemma 4.19. *Let $m \geq 2$ be an even integer and $n \geq 3$. We have*

$$\mathcal{L}(H_n^{m+1}) \cong \bigoplus_{x \in \mathbb{Z}_n/2} \mathcal{M}_x \ ,$$

where $\mathcal{M}_2 = M_{n^{m/2}}$. Furthermore, T_t leaves each $\mathcal{M}_x$ invariant.

Proof. Most of the argument utilizes the proof of Proposition 4.14 and [23, Lemma 5.3]. Note that $\lambda(\mathbb{Z}_n/2, 0)$ lives in the center of $\mathcal{L}(H_n^{m+1})$. By the decomposition of von Neumann algebras for the subalgebras of the center we obtain the first assertion. Write $e_j, j = 1, \cdots, m$ for the canonical basis of $\mathbb{Z}_n^m$ and put $u_r^j = \lambda(0, re_j)$ for $r \in \mathbb{Z}_n$. Then these u_r^j's generate $\lambda(0, \mathbb{Z}_n^m)$ and $u_r^j u_s^k(\delta_{(x,\cdot)}) = e^{2\pi i rs\theta_{jk}x/n}u_s^k u_r^j(\delta_{(x,\cdot)})$. Acting on $H_2 := \mathrm{span}\{\delta_{(2,\cdot)}\}$, u_r^j's satisfy $u_r^j u_s^k = e^{2\pi i rs/n}u_s^k u_r^j$ for $j < k$ and $u_r^j u_s^k = e^{-2\pi i rs/n}u_s^k u_r^j$ for $j > k$. It is clear that $y(r_1, \cdots, r_m) = u_{r_1}^1 \cdots u_{r_m}^m$ is a basis for $\mathcal{M}_2$ which satisfies the equation

$$u_r^j y(r_1, \cdots, r_m) u_r^{j*} \ = \ C(r, j, r_1, \cdots, r_m) y(r_1, \cdots, r_m) \ ,$$

where $C(r, j, r_1, \cdots, r_m) = \exp(2\pi i r(r_1 + \cdots + r_{j-1} - r_{j+1} - \cdots - r_m)/n)$. In order to determine the center of $\mathcal{M}_2$, we consider the equation for $C(r, j, r_1, \cdots, r_m) = 1$ for all $r \in \mathbb{Z}_n, j = 1, \cdots, m$. This leads to a linear system over $\mathbb{Z}_n$

$$\begin{aligned} -r_2 - \cdots - r_m \ &= \ 0, \\ r_1 - r_3 - \cdots - r_m \ &= \ 0, \\ \vdots \quad &\ \ \vdots \\ r_1 + \cdots + r_{m-1} \ &= \ 0. \end{aligned}$$

Solving this system, we find $r_1 = \cdots = r_m = 0$. Here we used the crucial assumption that m is even. Hence $\mathcal{M}_2$ has trivial center. Since it has dimension n^m, it follows that $\mathcal{M}_2 = M_{n^{m/2}}$, as desired. By restricting T_t to $M_{n^{m/2}}$ and repeating the argument of Proposition 4.14, we can prove the last assertion. $\square$

It follows from the lemma that $M_{n^{m/2}}$ admits a standard nc-diffusion semigroup T_t with $\Gamma_2 \geq \frac{n+2}{2n}\Gamma$ in $M_{n^{m/2}}$ for all $m \in 2\mathbb{N}$. Since the hyper finite II_1 factor R is the weak closure of $\cup_{k=1}^{\infty} M_{n^k}$, we have proved the following result.

Proposition 4.20. *For any integer $n \geq 2$, there exists a standard nc-diffusion semigroup T_t^n acting on R such that the associated gradient form Γ^n satisfies $\Gamma_2^n \geq \frac{n+2}{2n}\Gamma^n$ in R. The constant $\alpha_n = \frac{n+2}{2n}$ is best possible.*

The last conclusion follows from Proposition 4.12. Let us now recall some definitions from dynamical systems. Let $(X, \mathcal{B}_X, \mu, (T_t)_{t\geq 0})$ be a measure-preserving dynamic system (MPDS) where $\mathcal{B}(X)$ is a σ-algebra over X, μ is a probability measure and T_t is a semigroup of measure-preserving maps. A MPDS $(Y, \mathcal{B}_Y, \nu, (S_t)_{t\geq 0})$ is said to be a subsystem of $(X, \mathcal{B}_X, \mu, (T_t)_{t\geq 0})$ if there exist a measure-preserving measurable map $\pi : X \to Y$ and a number $a > 0$ such that for all $x \in X$, $\pi(T_t x) = S_{at}(\pi x)$ and for all $B \in \mathcal{B}_Y$, $\mu(\pi^{-1}B) = \nu(B)$. This motivates our following definition.

Definition 4.21. *Let S_t and T_t be standard semigroups acting on von Neumann algebras $\mathcal{N}$ and $\mathcal{M}$ respectively. Suppose $\mathcal{N} \subset \mathcal{M}$. $(\mathcal{M}, T_t)$ is said to be a dynamical subsystem of $(\mathcal{N}, S_t)$ if there exists a trace-preserving $*$-homomorphism $\pi : \mathcal{M} \to \mathcal{N}$ and $\lambda > 0$ such that $S_{\lambda t}(\pi x) = \pi(T_t x)$ for all $x \in \mathcal{M}$. We denote this by $(\mathcal{M}, T_t) \subset (\mathcal{N}, S_{\lambda t})$. $(\mathcal{M}, T_t)$ and $(\mathcal{N}, S_t)$ are isomorphic if they are subsystems of each other and π is a $*$-isomorphism.*

The following result shows that R admits infinitely many non-isomorphic standard nc-diffusion semigroups.

Proposition 4.22. *Let T_t^n be the semigroup considered in Proposition 4.20. If (R, T_t^n) and $(R, T_t^{n'})$ are isomorphic, then $\alpha_n = \alpha_{n'}$.*

Proof. There exists a trace-preserving $*$-isomorphism $\pi : R \to R$ such that

$$\pi(T_t^n x) = T_{\lambda t}^{n'}(\pi x) \tag{4.2}$$

for $x \in \mathcal{M}$. Let A^n be the generator of T_t^n. $\Gamma_2^{A^{n'}} \geq \frac{n'+2}{2n'}\Gamma^{A^{n'}}$ implies $\Gamma_2^{\lambda A^{n'}} \geq \lambda \frac{n'+2}{2n'}\Gamma^{\lambda A^{n'}}$. This together with (4.2) gives $\Gamma_2^{A^n} \geq \lambda \frac{n'+2}{2n'}\Gamma^{A^n}$. But the best α is $\alpha_n = \frac{n+2}{2n}$. Hence we have $\frac{n+2}{2n} \geq \lambda \frac{n'+2}{2n'}$. It is clear that $\mathrm{sp}(A^n) = \mathbb{N}$ and $\mathrm{sp}(\lambda A^n) = \lambda\mathbb{N}$. Here $\mathrm{sp}(A_n)$ denotes the spectrum of A_n. (4.2) implies $\mathrm{sp}(aA^n) = \mathrm{sp}(A^n)$ and thus $\lambda = 1$. Hence $n' \geq n$. Repeating the argument by starting from $\Gamma_2^n \geq \frac{n+2}{2n}\Gamma^n$ gives $n \geq n'$. □

5. Tensor products and free products

In this section we will construct further examples with the Γ_2-criterion based on the examples considered in the previous section. This is done via the powerful algebraic tools

– tensor products and free products. It is not difficult to see the property "standard nc-diffusion" is stable under tensor products and free products. Due to the reason explained in the previous section, it suffices to consider the algebraic Γ_2-condition. That is, we always work with a dense subalgebra contained in the domain of the form under consideration.

5.1. **Tensor products.** The following result is our starting point to understand tensor products.

Lemma 5.1. *Let $\Theta : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ and $\Phi : \mathcal{B} \times \mathcal{B} \to \mathcal{N}$ be positive sesquilinear forms, where $\mathcal{A} \subset \mathcal{M}$ and $\mathcal{B} \subset \mathcal{N}$ are dense subalgebras so that Θ and Φ are well-defined. Then $\Theta \otimes \Phi : \mathcal{A} \otimes \mathcal{B} \times \mathcal{A} \otimes \mathcal{B} \to \mathcal{M} \otimes \mathcal{N}$ is positive where for $\xi^i = \sum_{k=1}^{n_i} x_k^i \otimes y_k^i \in \mathcal{A} \otimes \mathcal{B}$,*

$$\Theta \otimes \Phi\Big(\xi^i, \xi^j\Big) := \sum_{k=1}^{n_i} \sum_{l=1}^{n_j} \Theta(x_k^i, x_l^j) \otimes \Phi(y_k^i, y_l^j) .$$

Proof. For $r \in \mathbb{N}$, let $(x_k^i) \subset \mathcal{A}, (y_k^i) \subset \mathcal{B}$ where $k = 1, \cdots, n_i, i = 1, \cdots, r$. Put $m = \sum_{i=1}^r n_i$. Without loss of generality we may assume $n_i = n$ for $i = 1, \cdots, r$. Suppose $\mathcal{M}$ and $\mathcal{N}$ act on Hilbert spaces H and K respectively. Then $(\Theta(x_k^i, x_l^j))_{k,l,i,j} = \sum_{i,j,k,l} \Theta(x_k^i, x_l^j) \otimes e_{(k,i),(l,j)} \geq 0$ as an operator on $\ell_2^m(H)$ where the indices $1 \leq k, l \leq n$. Similarly, $(\Phi(y_k^i, y_l^j))_{k,l,i,j} \geq 0$ on $\ell_2^m(K)$. It follows that

$$\begin{aligned} &(\Theta(x_k^i, x_l^j)) \otimes (\Phi(y_{k'}^{i'}, y_{l'}^{j'})) \\ = &\sum_{i,j,k,l,i',j',k',l'} \Theta(x_k^i, x_l^j) \otimes \Phi(y_{k'}^{i'}, y_{l'}^{j'}) \otimes e_{(k,i),(l,j)} \otimes e_{(k',i'),(l',j')} \ \geq \ 0 \end{aligned}$$

on $\ell_2^m(H) \otimes \ell_2^m(K)$. Define

$$v : \ell_2^m(H \otimes K) \to \ell_2^m(H) \otimes \ell_2^m(K), \quad \sum_s (\xi_t^s \otimes \eta_t^s) \otimes e_t \mapsto \sum_s (\xi_t^s \otimes e_t) \otimes (\eta_t^s \otimes e_t) .$$

Here $\xi_t^s \in H$, $\eta_t^s \in K$, and (e_t) is the canonical basis of ℓ_2^m for $t = 1, \cdots, m$. Then

$$v^* \Big[\sum_s (\xi_t^s \otimes e_t) \otimes (\eta_t^s \otimes e_t) \Big] = \sum_s (\xi_t^s \otimes \eta_t^s) \otimes e_t .$$

It is clear that $v^*[(\Theta(x_k^i, x_l^j)) \otimes (\Phi(y_{k'}^{i'}, y_{l'}^{j'}))]v \geq 0$. But

$$\begin{aligned} v^*[(\Theta(x_k^i, x_l^j)) \otimes (\Phi(y_{k'}^{i'}, y_{l'}^{j'}))]v \ &= \ \sum_{i,j,k,l} \Theta(x_k^i, x_l^j) \otimes \Phi(y_k^i, y_l^j) \otimes e_{(k,i),(l,j)} \\ &= \ [\Theta(x_k^i, x_l^j) \otimes \Phi(y_k^i, y_l^j)]_{(k,i),(l,j)} . \end{aligned}$$

Let $\mathbf{1}_n$ be a $n\times 1$ column vector with each entry equal to $1_{\mathcal{M}}\otimes 1_{\mathcal{N}}$ and I_r the $r\times r$ identity matrix. Define an operator $w:\ell_2^r(H\otimes K)\to \ell_2^m(H\otimes K)$ by

$$w \;=\; \mathbf{1}_n\otimes I_r \;=\; \begin{pmatrix} \mathbf{1}_n & & & \\ & \mathbf{1}_n & & \\ & & \ddots & \\ & & & \mathbf{1}_n \end{pmatrix} .$$

w is an $m\times r$ matrix. Note that $[\Theta(x_k^i,x_l^j)\otimes\Phi(y_k^i,y_l^j)]$ is an $m\times m$ matrix. Then

$$\begin{aligned} 0 \;&\le\; w^*\Big[\sum_{i,j,k,l}\Theta(x_k^i,x_l^j)\otimes\Phi(y_k^i,y_l^j)\otimes e_{(k,i),(l,j)}\Big]w \\ &=\; \sum_{i,j=1}^{r}\sum_{k=1}^{n_i}\sum_{l=1}^{n_j}\Theta(x_k^i,x_l^j)\otimes\Phi(y_k^i,y_l^j)\otimes e_{i,j} \\ &=\; \sum_{i,j=1}^{r}\Theta\otimes\Phi(\xi^i,\xi^j)\otimes e_{i,j}\ , \end{aligned}$$

which completes the proof. $\square$

Lemma 5.2. *Let $(T_t)_{t\ge 0}$ and $(S_t)_{t\ge 0}$ be standard semigroups with generator A and B acting on finite von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ respectively such that $\Gamma_2^A\ge\alpha\Gamma^A$ in $\mathcal{M}$ and $\Gamma_2^B\ge\alpha\Gamma^B$ in $\mathcal{N}$. Then $\Gamma_2^{A\otimes I}\ge\alpha\Gamma^{A\otimes I}$, $\Gamma_2^{I\otimes B}\ge\alpha\Gamma^{I\otimes B}$ and $\Gamma_2^{A\otimes I+I\otimes B}\ge\alpha\Gamma^{A\otimes I+I\otimes B}$ all in $\mathcal{M}\otimes\mathcal{N}$.*

Proof. The first inequality follows from Lemma 5.1 with $\Theta=\Gamma^A$ and $\Phi(y_1,y_2)=y_1^*y_2$. The second inequality can be shown similarly. For the last one, note that

$$\begin{aligned} &\Gamma_2^{A\otimes I+I\otimes B}-\alpha\Gamma^{A\otimes I+I\otimes B} \\ &=(\Gamma_2^{A\otimes I}-\alpha\Gamma^{A\otimes I})+(\Gamma_2^{I\otimes B}-\alpha\Gamma^{I\otimes B})+2\Gamma^A\otimes\Gamma^B\ . \end{aligned}$$

Then the first two inequalities and Lemma 5.1 with $\Theta=\Gamma^A$ and $\Phi=\Gamma^B$ yield the assertion. $\square$

Proposition 5.3. *Let A_j be self-adjoint generators of standard nc-diffusion semigroups $(T_t^{A_j})$ acting on $\mathcal{N}_j$ and $\Gamma_2^{A_j}\;\ge\;\alpha\Gamma^{A_j}$ respectively for $j=1,..,n$ with the same constant $\alpha>0$. Then the tensor product generator $\otimes A_j(x_1\otimes\cdots\otimes x_n)=\sum_j x_1\otimes\cdots\otimes x_{j-1}\otimes A_j(x_j)\otimes x_{j+1}\otimes\cdots\otimes x_n$ generates a standard nc-diffusion semigroup $(T_t^{\otimes A_j})$ with*

$$\Gamma_2^{\otimes A_j}\;\ge\;\alpha\Gamma^{\otimes A_j}\ .$$

Proof. Note that $T_t^{\otimes A_j}=T_t^{A_1}\otimes\cdots\otimes T_t^{A_n}$. Since $(T_t^{A_j})$ is a standard nc-diffusion semigroup for $j=1,\cdots,n$, so is $T_t^{\otimes A_j}$. We prove the Γ_2-condition by induction. The case $n=2$

follows from Lemma 5.2. The general case follows by induction and repeatedly invoking Lemma 5.1 to deal with "cross terms" like $\Gamma^{I\otimes\cdots A_i\cdots\otimes I}\otimes\Gamma^{I\otimes\cdots A_j\cdots\otimes I}$. $\square$

Example 5.4 (Tensor product of matrix algebras)**.** Let A be the generator of the semigroup T_t acting on M_n considered in Proposition 4.15. Let Γ be the gradient form associated to $\sum_{i=1}^m I\otimes\cdots\otimes A\otimes\cdots\otimes I$ where A is in the ith position. Then it follows from Proposition 5.3 that $\Gamma_2\ge\frac{2+n}{2n}\Gamma$ in $\otimes_{i=1}^m M_n$.

Example 5.5 (Random matrices)**.** Let $(\Omega,\mathbb{P})$ be a probability space. Consider $I\otimes T_t$ acting on $L_\infty(\Omega,\mathbb{P})\otimes M_n$ where T_t is the semigroup considered in Proposition 4.15. By Lemma 5.2, $I\otimes T_t$ is a standard nc-diffusion semigroup and satisfies $\Gamma_2\ge\frac{2+n}{2n}\Gamma$ in $L_\infty(\Omega,\mathbb{P})\otimes M_n$. Hence our results apply for random matrices.

Example 5.6 (Product measure)**.** Here we consider $A_j=I-E_j$ for E_j a conditional expectation on $\mathcal{N}_j$ for $j=1,\cdots,n$. By example 3.11, A_j generates a standard nc-diffusion semigroup and $\Gamma_2^{A_j}\ge\frac12\Gamma^{A_j}$. Then we deduce from Proposition 5.3 that $\Gamma_2^A\ge\frac12\Gamma^A$ for the tensor product generator $A=\otimes A_j$. For $x=x_1\otimes\cdots\otimes x_n$, put $\Gamma_j(x,x)=x_1^*x_1\otimes\cdots\otimes\Gamma^{A_j}(x_j,x_j)\otimes\cdots\otimes x_n^*x_n$. Then we have

$$\Gamma(x,x)\;=\;\sum_{j=1}^n\Gamma_j(x,x)\,.$$

We want to investigate an easy consequence of our general theory for the product measure space. Let $(\Omega_i,\mathbb{P}_i)$, $i=1,\cdots,n$ be a family of probability spaces and denote by $(\Omega,\mathbb{P})$ the product probability space. Then $L_\infty(\Omega,\mathbb{P})=\otimes_{i=1}^n L_\infty(\Omega_i,\mathbb{P}_i)$. Define $E_i(f)=\int fd\mathbb{P}_i$ for $f\in L_\infty(\Omega,\mathbb{P})$ and put $A_i=I-E_i$. Then

$$\begin{aligned}\Gamma_i(f,f)\;&=\;\frac12(|f|^2-f\int\bar f d\mathbb{P}_i-\bar f\int fd\mathbb{P}_i+\int|f|^2d\mathbb{P}_i)\\&=\;|f-\int fd\mathbb{P}_i|^2+\int\Big(|f|^2-\Big|\int fd\mathbb{P}_i\Big|^2\Big)d\mathbb{P}_i\,.\end{aligned}$$

It is straightforward to check that the fixed point subalgebra of the semigroup $e^{-t(\otimes A_i)}$ is $\mathbb{C}1$. Hence $E_{\mathrm{Fix}}f=\mathbb{E}f$ for $f\in L_\infty(\Omega,\mathbb{P})$ where $\mathbb{E}$ is the expectation operator of $\mathbb{P}$. Then (3.8) yields

$$\begin{aligned}\mathbb{P}(f-\mathbb{E}(f)\ge t)\;&\le\;\exp\Big(-\frac{ct^2}{\|\sum_{i=1}^n\Gamma_i(f,f)\|_\infty}\Big)\\&\le\exp\Big(-\frac{ct^2}{\sum_{i=1}^n\|f-\int fd\mathbb{P}_i\|_\infty^2+\|\int(|f|^2-|\int fd\mathbb{P}_i|^2)d\mathbb{P}_i\|_\infty}\Big)\,.\end{aligned}\tag{5.1}$$

Note that we do not impose any concrete condition on the probability spaces. This shows that the sub-Gaussian tail behavior is always true for product measures. We do not know whether such results are known before.

5.2. **Free products with amalgamation.** Here we want to prove that the condition $\Gamma_2 \ \geq\ \alpha\Gamma$ is stable under free products. Our general reference is [52]. We need some preliminary facts about free product of semigroups $T_t = *_k T_t^{A_k}$ acting on $\mathcal{N} := *_{\mathcal{D},k}\mathcal{N}_k$ with generators A_k acting on von Neumann algebra $\mathcal{N}_k \supset \mathcal{D}$. Here $\mathcal{D}$ is a von Neumann subalgebra of all $\mathcal{N}_k$. Similar to the tensor products considered before, if $(T_t^{A_k})$ is a standard nc-diffusion semigroup for $k = 1, \cdots, n$, so is $*_k T_t^{A_k}$. We assume that A_k commutes with the conditional expectation $E : \mathcal{N}_k \to \mathcal{D}$ for which we amalgamate and even

$$A_k E \ =\ E A_k \ =\ 0\,.$$

Our first task is to calculate the gradient Γ. For simplicity of notation, we always assume the elements we considered are chosen so that Γ and Γ_2 are well-defined. Let us now consider elementary words $x = a_1 \cdots a_m$ and $y = b_1 \cdots b_n$ of mean 0 elements $a_k \in \mathcal{N}_{i_k}$, $b_k \in \mathcal{N}_{j_k}$. Recall that the free product generator is given by

$$A(b_1 \cdots b_n) \ =\ \sum_{l=1}^{n} b_1 \cdots b_{l-1} A_{j_l}(b_l) b_{l+1} \cdots b_n\,.$$

In the future we will ignore the index for A. If we want to apply the free product generator A on the product of x^*y, we have to know the mean 0 decomposition

$$\begin{aligned} x^*y =& a_m^* \cdots a_1^* b_1 \cdots b_n \\ =& \sum_{k=1}^{\min(n,m)} a_m^* \cdots a_{k+1}^* \overbrace{a_k^* E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k}^{\circ}\, b_{k+1} \cdots b_n \\ &+ \begin{cases} E(a_m^* \cdots a_1^* b_1 \cdots b_m) b_{m+1} \cdots b_n, \text{if } m \leq n, \\ a_m^* \cdots a_{n+1}^* E(a_n^* \cdots a_1^* b_1 \cdots b_m), \text{if } m > n. \end{cases} \end{aligned}$$

Here $\mathring{x} = x - E(x)$. Let $k_0 = \inf\{i \in \mathbb{N} : k_i \neq j_i\}$. Since $T_t(E(x)) = E(x)$ implies that

$$A(E(x)y) \ =\ \lim_{t\to 0} \frac{T_t(E(x)y) - E(x)y}{t} \ =\ E(x)A(y) \tag{5.2}$$

it is easy to see that all terms containing $A(a_i^*)$, $A(b_i)$ for $i \geq k_0$ will cancel out in $\Gamma(x, y)$ and thus

$$\begin{aligned} 2\Gamma(x,y) \ =& \sum_{i=1}^{k_0-1} a_m^* \cdots a_{i+1}^* A(a_i^*) a_{i-1} \cdots a_1^* b_1 \cdots b_n + \sum_{i=1}^{k_0-1} a_m^* \cdots a_1^* b_1 \cdots b_{i-1} A(b_i) b_{i+1} \cdots b_n \\ & - a_m^* \cdots a_{k_0}^* A(a_{k_0-1}^* \cdots a_1^* b_1 \cdots b_{k_0-1}) b_{k_0} \cdots b_n \end{aligned}$$

$$= 2a_m^* \cdots a_{k_0}^* \Gamma(a_1 \cdots a_{k_0-1}, b_1 \cdots b_{k_0-1}) b_{k_0} \cdots b_n \, .$$

Lemma 5.7. *Let $a_i, b_i \in \mathcal{N}_{k_i}$ be mean 0 elements for $i = 1, \cdots, r$. Then*

$$\Gamma(a_1 \cdots a_r, b_1 \cdots b_r) \;=\; a_r^* \Gamma(a_1 \cdots a_{r-1}, b_1 \cdots b_{r-1}) b_r + \Gamma(a_r, E(a_{r-1}^* \cdots a_1^* b_1 \cdots b_{r-1}) b_r) \, .$$

Proof. Using the mean 0 decomposition, we have

$$\begin{aligned}
&2\Gamma(a_1 \cdots a_r, b_1 \cdots b_r) \;=\; A(a_r^*) a_{r-1}^* \cdots b_{r-1} b_r + a_r^* A(a_{r-1}^* \cdots a_1^*) b_1 \cdots b_r \\
&\quad + a_r^* \cdots a_1^* A(b_1 \cdots b_{r-1}) b_r + a_r^* \cdots b_{r-1} A(b_r) \\
&\quad - A(a_r^*) \Big(\sum_{i=1}^{r-1} a_{r-1}^* \cdots a_{i+1}^* \overbrace{a_i^* E(a_{i-1}^* \cdots a_1^* b_1 \cdots b_{i-1}) b_i}^{\circ} b_{i+1} \cdots b_{r-1} \Big) b_r \\
&\quad - a_r^* \Big(\sum_{i=1}^{r-1} a_{r-1}^* \cdots a_{i+1}^* \overbrace{a_i^* E(a_{i-1}^* \cdots a_1^* b_1 \cdots b_{i-1}) b_i}^{\circ} b_{i+1} \cdots b_{r-1} \Big) A(b_r) \\
&\quad - a_r^* A(a_{r-1}^* \cdots a_1^* b_1 \cdots b_{r-1}) b_r - A(a_r^* E(a_{r-1}^* \cdots b_{r-1}) b_r) \\
&= 2a_r^* \Gamma(a_1 \cdots a_{r-1}, b_1 \cdots b_{r-1}) b_r + A(a_r^*) E(a_{r-1}^* \cdots b_{r-1}) b_r \\
&\quad + a_r^* E(a_{r-1}^* \cdots b_{r-1}) A(b_r) - A(a_r^* E(a_{r-1}^* \cdots b_{r-1}) b_r)
\end{aligned}$$

which completes the proof with the help of (5.2). □

The recursion formula immediately yields that

Lemma 5.8. *Let k_0 be as above. Then*

$$\Gamma(x, y) \;=\; \sum_{k=1}^{k_0-1} a_m^* \cdots a_{k+1}^* \Gamma(a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k) b_{k+1} \cdots b_n \, .$$

Now we want to calculate Γ_2. For this we first define our new form

$$\Gamma^{(k)}(x, y) \;=\; a_m^* \cdots a_{k+1}^* \Gamma(a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k) b_{k+1} \cdots b_n \, .$$

We have to analyze

$$\Gamma^{(k)}(A(x), y) + \Gamma^{(k)}(x, A(y)) - A\Gamma^{(k)}(x, y) \, .$$

Observe that for $j < k$ all terms containing $A(a_j^*)$ or $A(b_j)$ appear inside the conditional expectation E in $\Gamma^{(k)}(A(x), y) + \Gamma^{(k)}(x, A(y))$ and there is no counterpart in $A\Gamma^{(k)}$. Hence we find

$$\begin{aligned}
I^{(k)}(x, y) \;&= a_m^* \cdots a_{k+1}^* \Gamma(a_k, E(A(a_{k-1}^* \cdots a_1^*) b_1 \cdots b_{k-1}) b_k) b_{k+1} \cdots b_n \\
&\quad + a_m^* \cdots a_{k+1}^* \Gamma(a_k, E(a_{k-1}^* \cdots a_1^* A(b_1 \cdots b_{k-1})) b_k) b_{k+1} \cdots b_n \, .
\end{aligned}$$

For $k \le j < k_0$ we are left with the following terms

$$\begin{aligned} II^{(k)}(x,y) \;=\; & \sum_{j=k}^{k_0-1} a_m^* \cdots A(a_j^*) \cdots a_{k+1}^* \Gamma(a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k) b_{k+1} \cdots b_n \\ & + \sum_{j=k}^{k_0-1} a_m^* \cdots a_{k+1}^* \Gamma(a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k) b_{k+1} \cdots A(b_j) \cdots b_n \\ & - a_m^* \cdots A(a_{k_0-1}^* \cdots a_{k+1}^* \Gamma(a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k) b_{k+1} \cdots b_{k_0-1}) \cdots b_n \end{aligned}$$

where we understand when $j = k$, $A(a_k)$ and $A(b_k)$ are inside the Γ-form. Since

$$\Gamma(a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k) \in \mathcal{N}_{i_k},$$

we are in the situation of Lemma 5.7. The recursion formula gives that

$$II^{(k)}(x,y) \;=\; \sum_{j=k+1}^{k_0-1} F_{jk}(x,y) + 2a_m^* \cdots a_{k+1}^* \Gamma_2(a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k) b_{k+1} \cdots b_n$$

where

$$\begin{aligned} F_{jk}(x,y) \;=\; & a_m^* \cdots a_{j+1}^* \Gamma(a_j, E(a_{j-1}^* \cdots \Gamma(a_k, \\ & E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k) \cdots b_{j-1}) b_j) b_{j+1} \cdots b_n \, . \end{aligned}$$

Therefore we find Γ_2.

Lemma 5.9. *Using the above notation, we have*

$$\begin{aligned} 2\Gamma_2(x,y) \;&=\; \sum_{k=1}^{k_0-1} \Gamma^{(k)}(A(x),y) + \Gamma^{(k)}(x,A(y)) - A\Gamma^{(k)}(x,y) \\ &=\; \sum_{k=1}^{k_0-1} I^{(k)}(x,y) + II^{(k)}(x,y) \end{aligned}$$

In order to show $\Gamma_2 \ge \alpha\Gamma$, we need a technical lemma which is an application of the Hilbert W^*-module theory; see [33, 39].

Lemma 5.10. *Let $\Phi : \mathcal{A} \times \mathcal{A} \to \mathcal{N}$ be a sesquilinear form where $\mathcal{A}$ is a separable *-algebra contained in the domain of Φ and $\mathcal{N}$ is a von Neumann algebra. Then Φ is a positive form if and only if there exists a map $v : \mathcal{A} \to C(\mathcal{N})$ such that $\Phi(x,y) = v(x)^* v(y)$ for $x, y \in \mathcal{A}$ where $C(\mathcal{N}) = \ell_2 \otimes \mathcal{N}$ denotes the Hilbert $\mathcal{N}$-module, or the column space of $\mathcal{N}$.*

Proof. The sufficiency is obvious. Conversely, following the KSGNS construction [33, 39], we consider the algebraic tensor product $\mathcal{A} \otimes \mathcal{N}$ and define $\langle \sum_i x_i \otimes a_i, \sum_j y_j \otimes b_j \rangle = \sum_{i,j} a_i^* \Phi(x_i, y_j) b_j$ for $x_i, y_j \in \mathcal{A}$ and $a_i, b_j \in \mathcal{N}$. Set $\mathcal{K} = \{x \in A \otimes \mathcal{N} : \langle x, x \rangle = 0\}$. Then

$\mathcal{A}\otimes\mathcal{N}/\mathcal{K}$ is a pre-Hilbert $\mathcal{N}$-module with $\mathcal{N}$-valued inner product $\langle x+\mathcal{K}, y+\mathcal{K}\rangle = \langle x, y\rangle$ for $x, y \in \mathcal{A}\otimes\mathcal{N}$. Let $\mathcal{A}\otimes_\Phi \mathcal{N}$ be the completion of $\mathcal{A}\otimes\mathcal{N}/\mathcal{K}$. Then $\mathcal{A}\otimes_\Phi\mathcal{N}$ is a Hilbert $\mathcal{N}$-module. Since $\mathcal{A}$ is separable, $\mathcal{A}\otimes_\Phi\mathcal{N}$ is countably generated. It follows from [33, Theorem 6.2] that there exists a right module map $u : \mathcal{A}\otimes_\Phi\mathcal{N} \to \ell_2\otimes\mathcal{N}$ such that

$$\sum_{i,j} a_i^*\Phi(x_i, y_j)b_j \;=\; \langle u(\sum_i x_i\otimes a_i + \mathcal{K}), u(\sum_j y_j\otimes b_j + \mathcal{K})\rangle\;.$$

In particular, $\Phi(x, y) = u(x\otimes 1+\mathcal{K})^*u(y\otimes 1+\mathcal{K})$. Define $v(x) = u(x\otimes 1+\mathcal{K})$. This is the desired map. □

Remark 5.11. Since we only consider finitely many elements for the sake of positivity of a form in the following, the separability assumption on $\mathcal{A}$ in the previous lemma is automatically satisfied. If we want to remove separability, we can use the fact that every Hilbert right module over $\mathcal{N}$ embeds isometrically in a self-dual module. Indeed, this follows from [39]. Let us sketch the approach from [28]. Consider the $L_{1/2}(\mathcal{N})$ module

$$Y = X\otimes_{\mathcal{N}} L_1(\mathcal{N}).$$

Then the antilinear dual Y^* is self-dual and obviously contains X isometrically.

By a similar argument of (5.2), we have $A^{1/2}(zx) = zA^{1/2}(x)$ for $z\in\mathcal{D}$ and $x\in\mathcal{N}$. Then for $x, y\in\mathcal{N}$,

$$\tau(zE(A(x)y)) = \tau(E(A(zx)y)) = \tau(A^{1/2}(zx)A^{1/2}(y)) = \tau(zE(A^{1/2}(x)A^{1/2}(y))).$$

Hence, $E(A(x)y) = E(A^{1/2}(x)A^{1/2}(y))$ and we find

$$I^{(k)}(x, y) \;=\; 2a_m^*\cdots\Gamma(a_k, E(A^{1/2}(a_{k-1}^*\cdots a_1^*)A^{1/2}(b_1\cdots b_{k-1}))b_k)b_{k+1}\cdots b_n\;.$$

We claim that this is a positive form. Indeed, I is nontrivial only if a_k and b_k are in the same $\mathcal{N}_{i_k}$. Using Lemma 5.10 with $\Phi = \Gamma$, we find $\beta_k : \mathcal{N}_{i_k}\to C(\mathcal{N}_{i_k})$ such that $\Gamma(a, b) = \beta_k(a)^*\beta_k(b)$ for $a, b\in\mathcal{N}_{i_k}$. Similarly with $\Phi(x, y) = E(x^*y)$, we find $v_k : \mathcal{N}\to C(\mathcal{D})$ such that $E(x^*y) = v_k(x)^*v_k(y)$ for $x, y\in\mathcal{N}$. Define

$$u_k(b_1\cdots b_n) = e_{i_1,\cdots,i_k}\otimes(\beta_k\otimes Id)(v_k(A^{1/2}(b_1\cdots b_{k-1}))b_k)b_{k+1}\cdots b_n\;.$$

Note that by the module property (5.2) $\Gamma(z^*a, b) = \Gamma(a, zb)$ for $z\in\mathcal{D}, x, y\in\mathcal{N}_{i_k}$. Write $v_k(x) = (v_k^l(x))_l$ where $v_k^l(x)\in\mathcal{D}$. It follows that

$$\begin{aligned}
I^{(k)}(x, y) \;&=\; 2a_m^*\cdots a_{k+1}^*\Gamma(a_k,\ \sum_l v_k^l[A^{1/2}(a_1\cdots a_{k-1})]^*v_k^l[A^{1/2}(b_1\cdots b_{k-1})]b_k)b_{k+1}\cdots b_n\\
&=\; 2\sum_l a_m^*\cdots a_{k+1}^*\Gamma(v_k^l[A^{1/2}(a_1\cdots a_{k-1})]a_k,\ v_k^l[A^{1/2}(b_1\cdots b_{k-1})]b_k)b_{k+1}\cdots b_n\\
&=\; 2\sum_l a_m^*\cdots a_{k+1}^*\beta_k(v_k^l[A^{1/2}(a_1\cdots a_{k-1})]a_k)^*\beta_k(v_k^l[A^{1/2}(b_1\cdots b_{k-1})]b_k)b_{k+1}\cdots b_n
\end{aligned}$$

$$= 2u_k(a_1\cdots a_m)^* u_k(b_1\cdots b_n)\ .$$

By Lemma 5.10, $I^{(k)}$ is a positive form.

Now we claim that F_{jk} are positive forms for $j = k+1,\cdots,k_0-1$. Indeed, define

$$\begin{aligned} u_{jk}(b_1\cdots b_n) &= e_{i_{k+1},\cdots,i_j}\otimes(\beta_j\otimes Id)((v_j\otimes Id)[e_{i_1,\cdots,i_k} \\ &\qquad \otimes(\beta_k\otimes Id)[v_k(b_1\cdots b_{k-1})b_k]b_{k+1}\cdots b_{j-1}]b_j)b_{j+1}\cdots b_n\ . \end{aligned}$$

Then similar to the argument for $I^{(k)}$, we find $F_{jk}(x,y) = u_{jk}(a_1\cdots a_m)^* u_{jk}(b_1\cdots b_n)$. By Lemma 5.10, F_{jk} is a positive form. Hence, we find

$$II^{(k)}(x,y) \ge 2a_m^*\cdots a_{k+1}^*\Gamma_2(a_k, E(a_{k-1}^*\cdots a_1^* b_1\cdots b_{k-1})b_k)b_{k+1}\cdots b_n\ .$$

Therefore we deduce the main result

Proposition 5.12. *Let A_j be self-adjoint generators of standard nc-diffusion semigroups $(T_t^{A_j})$ and $\Gamma^2_{A_j} \ \ge\ \alpha\Gamma_{A_j}$ respectively for $j = 1,..,n$ with the same constant $\alpha > 0$. Then the free product generator $*A_j(a_1\cdots a_n) = \sum_j a_1\cdots a_{j-1}A(a_j)a_{j+1}\cdots a_n$ generates a standard nc-diffusion semigroup $(T_t^{*A_j})$ with*

$$\Gamma^2_{*A_j}\ \ge\ \alpha\Gamma_{*A_j}\ .$$

Example 5.13. The free product of all the examples considered so far satisfies the Γ_2-criterion. In particular, the free product of matrix algebra $*_i M_n$ admits a standard nc-diffusion semigroup with the Γ_2-criterion.

Example 5.14 (Block length function)**.** Consider the free product of groups G_i, $G = *_{i\in I}G_i$ with block length function. Fix i and denote by λ the left regular representation of G_i. Define the conditional expectation $E : \mathcal{L}(G_i) \to \mathbb{C}1$ to be

$$E(\lambda(g))\ =\ \tau(\lambda(g))1\ =\ \begin{cases} 1, \text{ if } g = e, \\ 0, \text{ if } g \ne e. \end{cases}$$

Here e is the identity element of G_i and 1 is the identity element of $\mathcal{L}(G_i)$. Example 3.11 says that $T_t\lambda(g) = e^{-t(I-E)}\lambda(g)$ is a standard nc-diffusion semigroup with $\Gamma_2 \ge \frac{1}{2}\Gamma$ where

$$T_t\lambda(g)\ =\ \begin{cases} \lambda(g), \text{ if } g = e, \\ e^{-t}\lambda(g), \text{ if } g \ne e. \end{cases}$$

Since $\mathcal{L}(G) = *_{i\in I}\mathcal{L}(G_i)$, using Proposition 5.12 and the relation $\lambda(g_1\cdots g_n) = \lambda(g_1)\cdots\lambda(g_n)$ for $g_1 \in G_{i_1},\cdots,g_n \in G_{i_n}$ and $i_1 \ne i_2 \ne \cdots \ne i_n$, we deduce that (T_t^b) is a standard nc-diffusion semigroup acting on $\mathcal{L}(G)$ with $\Gamma_2 \ge \frac{1}{2}\Gamma$ where

$$T_t^b(\lambda(g_1^{k_1}\cdots g_n^{k_n})) = e^{-tn}\lambda(g_1^{k_1}\cdots g_n^{k_n}),\quad g_1^{k_1}\cdots g_n^{k_n} \in \mathbb{F}_I \text{ freely reduced.}$$

Clearly, the infinitesimal generator of T_t^b is the block length function. In particular, for $G_i = \mathbb{Z}$ we find a standard nc-diffusion semigroup acting on $\mathcal{L}(\mathbb{F}_n)$ which is different from the one considered in Section 4.1. In fact, our result applies even for free product of groups with amalgamation in general.

6. The classical diffusion processes

We consider classical diffusion semigroups in this section. As explained in Remark 3.9, we have stronger results in the this setting thanks to the better constant in the commutative BDG inequality.

6.1. **Ornstein-Uhlenbeck process in $\mathbb{R}^d$.** Let us start with Ornstein-Uhlenbeck process whose infinitesimal generator is $-A = \Delta - x \cdot \nabla$ in $\mathbb{R}^d$. We refer the readers to e.g. [34] for the facts we state in this subsection. Let $T_t = e^{-tA}$ be the semigroup generated by A and γ denote the canonical Gaussian measure on $\mathbb{R}^d$ with density $(2\pi)^{-d/2}e^{-|x|^2/2}$. It is well-known that γ is an invariant measure of T_t and

$$T_t f(x) = \int_{\mathbb{R}^d} f(e^{-t}x + (1-e^{-2t})^{1/2}y)d\gamma(y).$$

Let $\mathcal{A} = C_c^\infty(\mathbb{R}^d)$, the compactly supported smooth functions. Clearly $\mathcal{A}$ is weakly dense in $\mathcal{N} = L_\infty(\mathbb{R}^d, \gamma)$ and T_t is self-adjoint with respect to γ. Clearly $\mathcal{A}$ is dense in $\mathrm{Dom}(A^{1/2})$ in the graph norm. Note that $\Gamma(f,f) = |\nabla f|^2$ and that for $f \in \mathrm{Dom}(A^{1/2})$

$$\|\Gamma(f,f)\|_1 \;=\; \langle A^{1/2}f, A^{1/2}f\rangle_{L_2(\mathbb{R}^d,\gamma)} \,.$$

Therefore (T_t) is a standard nc-diffusion semigroup satisfying the assumptions in Lemma 3.6. It is easy to check that

$$\Gamma_2(f,f) = |\nabla f|^2 + \|\operatorname{Hess} f\|_{HS}^2 \geq \Gamma(f,f), \quad f \in C_c^\infty(\mathbb{R}^d).$$

Here $\operatorname{Hess} f$ denotes the Hessian of f and $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. Note that $Af = 0$ only if f is a constant. Thus the fixed point algebra is trivial. Theorem 3.4 with $\alpha = 1$ and Remark 3.9 immediately lead to the following result.

Corollary 6.1. *Let $2 \leq p < \infty$. Then there exist a constant C such that for all real valued functions $f \in W^{1,p}(\mathbb{R}^d, \gamma)$*

$$\left\| f - \int f d\gamma \right\|_{L_p(\mathbb{R}^d,\gamma)} \;\leq\; C\sqrt{p}\|\, |\nabla f|\, \|_{L_p(\mathbb{R}^d,\gamma)} \tag{6.1}$$

where $W^{1,p}(\mathbb{R}^d, \gamma)$ denotes the Sobolev space consisting of all $L_p(\mathbb{R}^d, \gamma)$ functions with first order weak derivatives also in $L_p(\mathbb{R}^d, \gamma)$.

This result can be generalized to infinite dimension. Let (W, H, μ) be an abstract Wiener space and L the Ornstein-Uhlenbeck operator on W. Then it can be checked that the gradient form associated with L satisfies

$$\Gamma_2(F, F) \;=\; (\nabla F, \nabla F) + \|\nabla^2 F\|_{HS}^2 \;\geq\; \Gamma(F, F)$$

for $F(w) \in \mathrm{Cylin}(W)$, the cylindrical functions on W. Based on standard facts from Malliavin calculus, similar argument to the $\mathbb{R}^d$ case shows that the Ornstein-Uhlenbeck semigroup T_t is a standard nc-diffusion semigroup satisfying the assumptions in Lemma 3.6. Moreover, the fixed point algebra Fix is trivial. See [17, 37] for more details. Hence our Poincaré type inequality (6.1) holds in this setting.

6.2. **Diffusion processes on Riemannian manifolds.** Consider an elliptic differential operator $-A$ on a connected smooth manifold M of dimension d with probability measure μ on Borel sets which is equivalent to Lebesgue measure. We can write it in a local coordinate chart as

$$-Af(x) = \sum_{i,j} g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_i b^i \frac{\partial f}{\partial x^i}(x)$$

where g^{ij} and b^i are smooth functions and (g^{ij}) is a nonnegative definite matrix. The inverse of (g^{ij}) then defines a Riemannian metric. It can be checked that

$$\Gamma(f, h) = \sum_{ij} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial h}{\partial x^j}$$

for all $f, h \in C_c^\infty(M)$. To give an example, we take $-A = \Delta + Z$ where Δ is the Laplace-Beltrami operator on a complete Riemannian manifold and Z is a C^1-vector filed Z on a Riemannian manifold M such that

$$\mathrm{Ric}(X, X) - \langle \nabla_X Z, X\rangle \geq \alpha |X|^2, \quad X \in TM \tag{6.2}$$

for some $\alpha > 0$. By the Bochner identity, this inequality is equivalent to (see e.g. [53])

$$\Gamma_2(f, f) \geq \alpha \Gamma(f, f), \quad f \in C^\infty(M).$$

Take $\mathcal{A} = C_c^\infty(M)$ and $T_t = e^{-tA}$. The following result follows from Remark 3.9 and the martingale problem on differential manifolds [20].

Corollary 6.2. *Assume* (6.2) *and the following conditions*

(1) $\int T_t(f) g d\mu = \int f T_t(g) d\mu$ *(i.e. T_t is symmetric);*
(2) $|\nabla f| \in L_2(M, \mu)$ *whenever* $\langle A^{1/2} f, A^{1/2} f\rangle < \infty$.

Then for all $2 \le p < \infty$ *and real valued functions* $f \in W^{1,p}(M,\mu)$,

$$\|f - E_{\mathrm{Fix}} f\|_{L_p(M,\mu)} \ \le\ C\sqrt{p/\alpha}\| \ |\nabla f| \ \|_{L_p(M,\mu)} \tag{6.3}$$

where $W^{1,p}(M,\mu)$ *is the Sobolev space on the Riemannian manifold* M.

Functional inequalities related to diffusion processes on Riemannian manifolds have been studied extensively; see [54] for more details on this subject. To give an even more concrete example, let ν be the normalized volume measure and $\mu(dx) = e^{-V(x)}\nu(dx)$ a probability measure for $V \in C^2(M)$. Suppose (6.2) holds. It is clear that the semigroup T_t with generator $-A = \Delta - \nabla V \cdot \nabla$ fulfills the assumptions of Corollary 6.2 and the fixed point algebra is trivial. It follows that

$$\|f - \int f d\mu\|_{L_p(M,\mu)} \ \le\ C\sqrt{p/\alpha}\| \ |\nabla f| \ \|_{L_p(M,\mu)}\ .$$

This improves X.-D. Li's result [35, Theorem 1.2, Theorem 5.2] for $p \ge 2$ which was proved by using his sharp estimate of the L_p-norm of Riesz transform. Indeed, his Poincaré inequality has constant $p/\sqrt{\alpha}$.

Remark 6.3. (6.3) is true only for scalar-valued functions. If one is interested in some noncommutative objects, e.g., matrix-valued functions on manifolds or free product of manifolds, one has to apply the noncommutative theory and then the Poincaré inequalities are in the form of Theorem 3.4. Of course, the deviation and the transportation inequalities still hold in all those situations.

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